

# NOTES ON THE DOMAIN OF EXPONENT PAIRS

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**ABSTRACT.** The theory of exponent pairs as initiated by Phillipps in 1933 proposes pairs of exponents  $(\kappa, \lambda)$  so that one has  $\sum_{n \sim N} e^{2i\pi\varphi(n)} \ll_\varepsilon F^{\kappa+\varepsilon} N^{\lambda+\varepsilon}$ , for any positive  $\varepsilon$ , where  $\varphi$  is a 'monomial-like' smooth function whose first derivative is of size about  $F$ . We propose to explore the domain of available pairs  $(\kappa, \lambda)$  through a very geometrical approach. We prove in particular that this domain is the convex hull of a connected curve in the classical case. We also show that a possible choice for  $\lambda$ , for any  $\kappa \in [0, 1/2]$ , is given by  $\lambda = 1 - \frac{\kappa}{\log 2} \log \frac{2\kappa+1}{2\kappa}$ . We finally recall rapidly how this theory has been adapted to the higher dimensional setting. In passing, we take the opportunity of this slow-paced paper to describe some usage of the SageMath software.

## 1. INTRODUCTION AND RESULTS

*Exponent pairs in the large.* E. Phillipps developed in [8] a theory of *exponent pairs* by furthering and simplifying the notion of *exponent system* introduced by J.G. van der Corput in [12]. The reader will find a modern account of this theory in the reference book [6] by S.W. Graham and G. Kolesnik. Roughly speaking a couple  $(\kappa, \lambda) \in [0, \frac{1}{2}] \times [\frac{1}{2}, 1]$  is said to be an *exponent pair* when, given a regular function  $\varphi$  that is 'monomial-like' and whose first derivative on the interval  $[N, 2N]$  is of size  $F$ , the upper bound

$$S = \sum_{N < n \leq 2N} e^{2i\pi\varphi(n)} \ll_\varepsilon F^{\kappa+\varepsilon} N^{\lambda+\varepsilon}.$$

holds for any  $\varepsilon > 0$ . The following exponent pairs are known:

$$(1) \quad (0, 1), \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{6}, \frac{2}{3}\right), \left(\frac{1}{14}, \frac{11}{14}\right), \left(\frac{9}{56}, \frac{37}{56}\right), \left(\frac{89}{560}, \frac{369}{560}\right).$$

*An example.* In [2], Dekking and Mendès-France propose a geometrical approach to exponential sums. The reader will find there several examples and some compelling drawings. A most classical example is  $\varphi(n) = t(\log n)/(2\pi)$  where  $t$  is some large parameter and for instance  $N = t^{1/3}$ . The sum  $S$  is then often called a *zeta-sum*. A first trivial bound for  $S$  is  $N + 1$ . Since  $\varphi'(n) = t/(2\pi n)$ , we see that  $t^{2/3} < 4\pi\varphi'(n) \leq 2t^{2/3}$ , so that we may select  $F = N^2$ . We thus get the bound  $S \ll_\varepsilon N^{2\kappa+\lambda+\varepsilon}$ , and with the pairs given in (1) above, this gives the exponents

$$(2) \quad 1, \frac{3}{2}, 1, \frac{13}{14} = 0.928 \dots, \frac{55}{56} = 0.982 \dots, \frac{547}{560} = 0.976 \dots$$

of which  $13/14$  is the best one for our problem.

We encourage the readers to find the best bound obtainable in shorter ranges, and for instance when  $N = t^{1/5}$ .

*The domain of exponent pairs.* It follows from the theory that, given an exponent pair  $(\kappa, \lambda)$ , we may build another one by the two formulas

$$(3) \quad \left(\frac{\kappa}{2\kappa+2}, \frac{\kappa+\lambda+1}{2\kappa+2}\right), \quad \left(\lambda - \frac{1}{2}, \kappa + \frac{1}{2}\right).$$

The first five pairs above are obtained by using these two processi while the last two have been obtained respectively by M.N. Huxley and N. Watt in [7] and by N. Watt in [13]. Furthermore, any convex combination of exponent pairs is again an exponent pair. When using the pair  $(0, 1)$  and the processi described in (3), we call the convex hull of the domain obtained the *van der Corput Domain* denoted by  $\mathcal{D}$ .

**Theorem 1.1.** *The domain  $\mathcal{D}$  is the convex hull of the curve  $\mathcal{C}$  defined in (9) and which is the graph of a continuous non-increasing function.*

When we add the point  $(\frac{89}{560}, \frac{369}{560})$ , the domain will be called the *Watt Domain* in the sequel and denoted by  $\mathcal{D}^*$ .

In [9], R.A. Rankin started to describe the set of accessible exponent pairs, a study furthered by S.W. Graham in [5]. The viewpoint taken in both papers is to compute optimal values in a specific problem. The aim of the present note is to continue this work from a more geometric viewpoint. Nonetheless it is fair to say that a large part of the material we present here can be found in the previous two papers in some form or some other.

Since we also strive to describe the situation with pictures, it may be better to provide the readers with the means to play themselves with these pictures. We shall be using SageMath, see [11]; the script we use is available on the web at:

<https://ramare-olivier.github.io/Maths/ExpPairsNote-01.sage>

Copy this code is a file named, say, `ExpPairs.sage`, without forgetting the `sage` suffix, start SageMath and load this via the command `load("ExpPairs.sage")`. We give in the text some pointers on to how to code in SageMath, as well as commands that we write in the form

`ExpPairs.sage/plotC(12, 6)[1]`

to mean that the reader should type the command `plotC(12, 6)[1]` in SageMath, once the main file `ExpPairs.sage` has been duly loaded.

*A simple continuous bound.* A consequence of our study is the next flexible estimate.

**Theorem 1.2.** *Let  $S$  be an exponential sum of monomial type and parameters  $N$  and  $F$ . Then, for every  $\kappa \in [0, \frac{1}{2}]$  and every  $\varepsilon > 0$ , we have  $S \ll_\varepsilon F^{\kappa+\varepsilon} N^{\vartheta_0(\kappa)+\varepsilon}$  where*

$$\vartheta_0(\kappa) = 1 - \frac{\kappa}{\log 2} \log \frac{2\kappa+1}{2\kappa}.$$

We have  $\vartheta_0(0) = 1$ ,  $\vartheta_0(1/2) = 1/2$  and  $\vartheta_0(1/6) = 2/3$ .

This is proved in Lemma 2.8 below. The upper bound  $S \ll_\varepsilon F^{\varphi(\lambda+\frac{1}{2})-\frac{1}{2}+\varepsilon} N^{\lambda+\varepsilon}$  also holds true for every  $\lambda \in [\frac{1}{2}, 1]$  and every  $\varepsilon > 0$ , and this one is better than the above one when  $\lambda \leq 2/3$ . Theorem 5.1 belows offers a generalization of this result to higher dimensional exponential sums. Contrarily to S.W. Graham's approach that leads to optimal values at some specific points, Theorem 1.2 allows real-valued optimization at a small numerical loss in the exponent. The approximation is however very tight as shown by Figure 1.

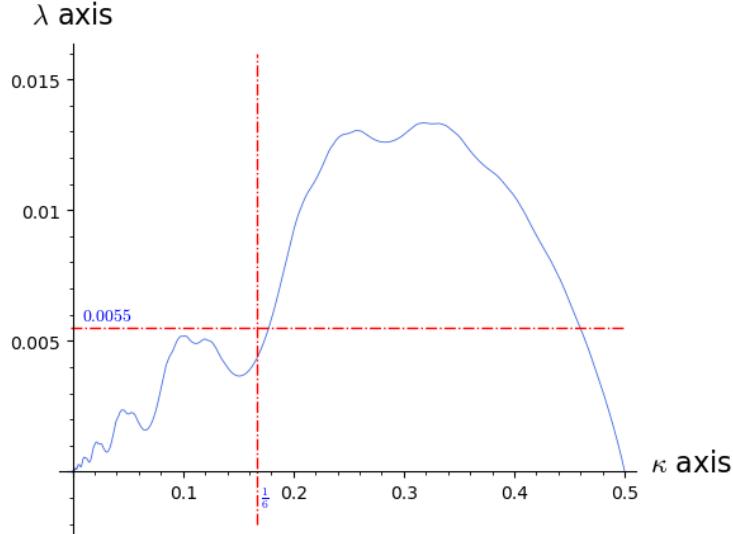


FIGURE 1. Difference between  $\vartheta_0$  and the optimal exponent pair from the van der Corput domain, drawing obtained via `ExpPairs.sage/compareModelConvHull(12)`.

On our example, we get  $S \ll_\varepsilon N^{2\kappa + \vartheta_0(\kappa) + \varepsilon}$ , which is minimal when  $\kappa = 0.0566\ 865\cdots$  with value  $N^{0.926\cdots}$ , whence

$$(4) \quad \sum_{t^{1/3} < n \leq 2t^{1/3}} n^{it} \ll t^{0.926/3}$$

improving on (2), though still far from the expected  $t^{\frac{1}{6} + \varepsilon}$ .

**Preparing for the proofs: a change of variables.** We prefer to change of variables and to use

$$(5) \quad (u, v) = (2\kappa, 2\lambda - 1).$$

The pairs above become  $(0, 1)$ ,  $(1, 0)$ ,  $(\frac{1}{3}, \frac{1}{3})$ ,  $(\frac{1}{7}, \frac{4}{7})$ ,  $(\frac{9}{28}, \frac{9}{28})$ ,  $(\frac{89}{280}, \frac{89}{280})$ , while the transformations (3) read

$$(6) \quad f(u, v) = \left( \frac{u}{u+2}, \frac{v+1}{u+2} \right), \quad c(u, v) = (v, u).$$

As  $c$  is an involution, it is better to consider the transform  $g = c \circ f$  and to consider iterations of  $f$  and  $g$ . It is noteworthy that  $f$  and  $g$  preserve segments. One can consider these transforms as restrictions of linear transforms on the projective plane  $\mathbb{P}_2$ :

$$(7) \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

so that  $f(u, v)$  can be read on the first two coordinates of  $C \begin{pmatrix} u \\ v \\ 1 \end{pmatrix}$  (when we divide by the third one); a similar link holds between  $g$  and  $D$ .

2. A FIRST PLAYER: THE CURVE  $\mathcal{C}$ 

There are two ways to describe the van der Corput Domain. Both rest on a curve  $\mathcal{C}$  that we now build: either by taking a limit from above, or by taking the closure of the set of points obtained by iterating  $f$  and  $g$  when starting from  $\{(0, 1), (1, 0)\}$ .

*Getting to  $\mathcal{C}$  from outside.* The construction we now describe will for instance make clear that we reach a connected curve.

In this section, we consider the transformation of the unit square  $[0, 1]^2$  under the two transforms  $f$  and  $g$ . Let us first notice that these transforms are contracting.

**Lemma 2.1.** *When  $P, Q \in [0, 1]^2$ , we have  $\|f(P) - f(Q)\|_2 \leq \rho\|P - Q\|_2$  where  $\rho = \sqrt{\frac{3+\sqrt{5}}{8}} \leq 13/16$ . The same holds true for  $g$ .*

*Proof.* The Jacobian reads, with  $U = u + 2$  and  $V = v + 1$ ,

$$J = \begin{pmatrix} \frac{2}{U^2} & -\frac{V}{U^2} \\ 0 & \frac{1}{U} \end{pmatrix} \quad \text{so that} \quad U^4 J J^* = \begin{pmatrix} 4 + V^2 & -UV \\ -UV & U^2 \end{pmatrix}.$$

The largest eigenvalue of  $J J^*$  is

$$\lambda = \frac{4 + U^2 + V^2 + \sqrt{U^4 + (2V^2 - 8)U^2 + (4 + V^2)^2}}{2U^4}.$$

It is largest when  $V = 2$ , so we are left with finding the maximum over  $W \in [4, 9]$  of the quantity

$$\frac{8 + W + \sqrt{64 + W^2}}{2W^2}.$$

As this function of  $W$  is non-increasing, the worst case is  $W = 4$ . The lemma follows readily.  $\square$

Let  $\mathcal{K}([0, 1]^2)$  be the compact space of the compact subsets of  $[0, 1]^2$ , equipped with the usual Hausdorff distance (see for instance Exercise 3 of Section 16, Chapter 3 of the reference book [3] by J. Dieudonné), i.e.

$$(8) \quad d(K_1, K_2) = \max \left( \max_{k_1 \in K_1} d(k_1, K_2), \max_{k_2 \in K_2} d(k_2, K_1) \right).$$

We, rather obviously, still call  $f$  the function induced by  $f$  on  $\mathcal{K}$ . And  $f \cup g$  is the function that, to any set  $\mathcal{A}$ , associates  $f(\mathcal{A}) \cup g(\mathcal{A})$ . This is a continuous function. We set

$$(9) \quad \mathcal{C} = \bigcap_{n \geq 0} (f \cup g)^{\circ n}([0, 1]^2)$$

where notation  $(f \cup g)^{\circ n}$  means that we compose  $(f \cup g)$  iteratively  $n$  times with itself. This set  $\mathcal{C}$  corresponds to the set  $\mathcal{P}$  of S.W. Graham in [5]. Plotting  $\mathcal{C}$  is not difficult.

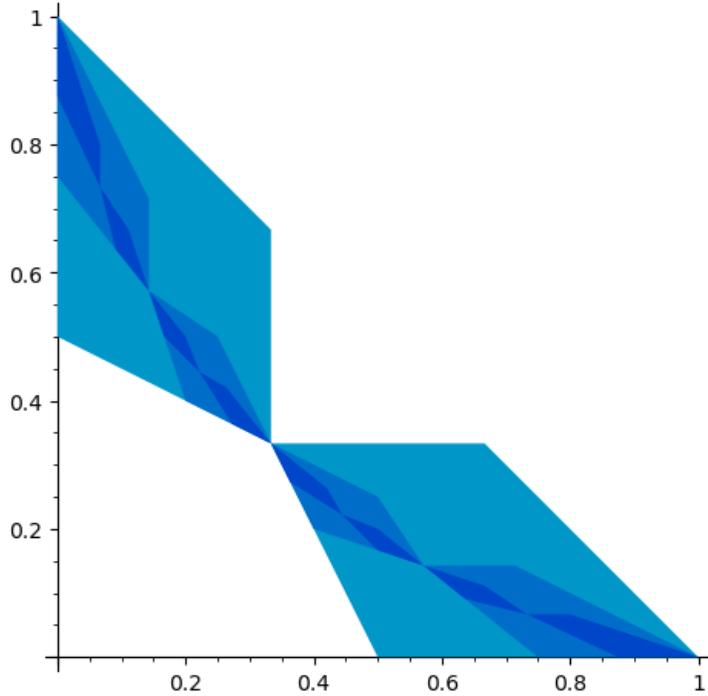


FIGURE 2. The first three iterations, drawing obtained via  
`ExpPairs.sage/transformSquare([[0,0],[0,1],[1,1],[1,0]],3,2,False)`.

Here are some details on the SageMath code that produces it.

```
#####
# Common handlers #####
def fC(p):
    return([p[0]/(p[0]+2), (p[1]+1)/(p[0]+2)])

def gD(p):
    return([(p[1]+1)/(p[0]+2), p[0]/(p[0]+2)])

#####
# Building of the C-curve from up #####
def actonD(InitDom, nbsteps, myz, doplot = True, shade = 0):
    # When doplot = False, be sure nbsteps > 0
    acolor = Color(0/255, (70 + 40*(nbsteps + shade))/255, 200/255)
    DomfC = list(map(fC, InitDomain))
    DomgD = list(map(gD, InitDomain))
    ToPlot = Graphics() # empty graphical object
    if doplot:
        ToPlot = polygon(InitDom, color = acolor, zorder = myz)
        if nbsteps > 0:
            ToPlot += actonD(DomfC, nbsteps-1, myz+1, True, shade)
            ToPlot += actonD(DomgD, nbsteps-1, myz+1, True, shade)
    else:
        if nbsteps > 0:
            ToPlot = actonD(DomfC, nbsteps-1, myz+1, True, shade)
            ToPlot += actonD(DomgD, nbsteps-1, myz+1, True, shade)
    return(ToPlot)
```

```
MyInitialDomain = [[0,0], [0,1], [1,1], [1,0]]
actonD(MyInitialDomain, 6, 2, False, 110).show(figsize = 15,
      gridlines = "automatic", xmin = 0, xmax = 1,
      ymin = 0, ymax = 1, aspect_ratio = 1)
```

Copy this code is a file `CCurve.sage`, without forgetting the `sage` suffix, start SageMath and run the file by using the command `load("CCurve.sage")`. It is then possible to increase `nbsteps`, say to 10, and to zoom on a particular region by changing the quadruple `(xmin, xmax, ymin, ymax)`. Let us take the opportunity of this note to explain part of the help system of Sagemath. If we set `P = actonD(MyInitialDomain, 1, 1, False, 0)`, then we may use `P.<tab>` to get access to all the methods associated with the object `P` (a plot). And to see all the tons of options associated with the `show` method, enter `P.show?`. The 2D-plotting reference guide is available there:

<https://doc.sagemath.org/pdf/en/reference/plotting/plotting.pdf>

Let us comment on this picture. Introducing an adhoc definition will simplify our task.

**Definition 2.2.** *A tile is the convex hull of four points.*

Since the functions  $f$  and  $g$  transform segments into segments, tiles are transformed into tiles by any composition-product of these two.

We start from the tile  $[0, 1]^2$ , which we transform by  $f$ , getting the new tile  $\text{Conv}((0, 1), (\frac{1}{3}, \frac{2}{3}), (\frac{1}{3}, \frac{1}{3}), (0, \frac{1}{2}))$  and then similarly by  $g$ , getting the tile  $\text{Conv}((\frac{1}{3}, \frac{1}{3}), (\frac{2}{3}, \frac{1}{3}), (1, 0), (\frac{1}{2}, 0))$ . These resulting tiles join in  $(\frac{1}{3}, \frac{1}{3})$ . On applying repeatedly the transforms  $f$  and  $g$ , we get a connected necklace of tiles. Here is a lemma that helps structure the situation.

**Lemma 2.3.** *Let  $K$  be a connected compact subset of  $[0, 1]^2$  that contains the points  $(0, 1)$  and  $(1, 0)$ . The set  $f(K) \cup g(K)$  is again a connected compact subset of  $[0, 1]^2$  that contains the points  $(0, 1)$  and  $(1, 0)$ .*

*Proof.* Indeed  $f(K)$  and  $g(K)$  are both connected and compact. Both sets contain the point  $f((1, 0)) = (\frac{1}{3}, \frac{1}{3}) = g((0, 1))$ , so that  $f(K) \cup g(K)$  is connected. It contains the points  $f((0, 1)) = (0, 1)$  and  $g((0, 1)) = (1, 0)$ .  $\square$

At each step, we get a succession of tiles  $h(0, 1) - h(1, 1) - h(1, 0) - h(0, 0)$ ; the distance between  $h(1, 1)$  and  $h(0, 0)$  is at most  $\rho^n \sqrt{2}$ , when  $h$  is a product of  $n$  terms from  $\{f, g\}$ . This shows an exponential rate, and the actual rate is faster (meaning the practical ' $\rho$ ' is smaller). One shows readily that we end up with a curve. To plot it, we may only consider the transforms of the lower part of the initial square. Let us state formally a theoretical consequence of this discussion.

**Lemma 2.4.** *The curve  $\mathcal{C}$  is the graph of a continuous non-increasing function.*

We shall now see that this seemingly regular curve  $\mathcal{C}$  contains a dense subset of *rational* points (i.e. points whose coordinates are rational numbers).

*Getting to  $\mathcal{C}$  from inside, I.* We may get points that are *on* the final curve by two processi. Here is a first one.

The construction above shows also that each point of  $\mathcal{C}$  may be reached in a unique manner either from any point with a *infinite* sequence of  $f$  and  $g$ , giving an adapted 'binary' writing for these points. For instance the point  $(1, 0)$  is  $fff\cdots$ , while  $(0, 1)$  is  $ggg\cdots$ : indeed, the first  $f$  reduces the unit square  $[0, 1]^2$  to a smaller parallelepiped that lies inside  $[0, 1/3] \times [1/3, 1]$ . On applying  $f$  again, we get an even smaller parallelepiped that still contains the point  $(0, 1)$ . The intersection

$$\bigcap_{n \geq 1} f^{\circ n}([0, 1]^2)$$

then reduces to the point  $(0, 1)$ . Please notice that when we associate the sequence  $fff\cdots$  to this points, the order is reverse to the one we use for the composition of functions.

The point  $(\frac{1}{3}, \frac{1}{3})$  is  $fffff\cdots = fgggg\cdots$ . The same construction shows that any finite combination of  $f$  and  $g$  applied to  $(0, 1)$  or  $(1, 0)$  belongs to  $\mathcal{C}$ . We get rational points by considering a finite sequence, say  $ffgfg$ , and completing it on the right either by  $fff\cdots$  if we want to refer to the upper left point of the parallelepiped  $gfgff([0, 1]^2)$ , or by  $ggg\cdots$  if we want to refer to the lower right point of the same parallelepiped. We get in this manner rational points that are on the curve  $\mathcal{C}$ . The reader will readily see that we get a dense family of such points. Indeed, specifying a prefix, like  $ffgfg$ , localizes the point inside  $gfgff([0, 1]^2)$  and any continuation, say  $ffgfgffggffgg$ , leads to points that are inside this set.

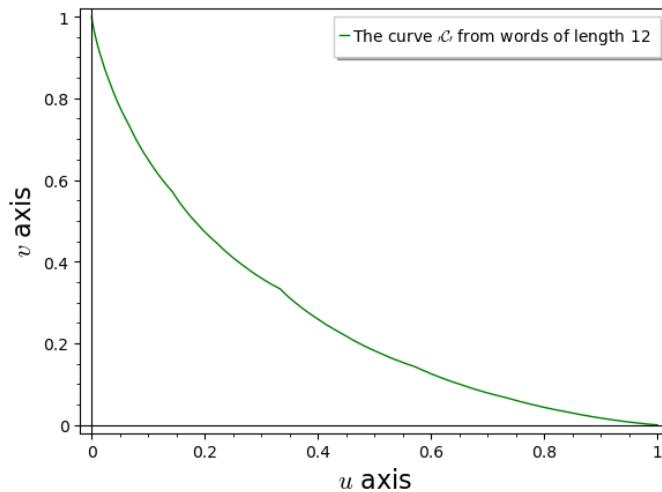


FIGURE 3. Approximation of the curve  $\mathcal{C}$  with words of length 12, drawing obtained through `ExpPairs.sage/plotC(12, 6)` [1].

*Getting to  $\mathcal{C}$  from inside, II.* We now describe a second process to get points that are *on* the final curve.

A point of the curve may be attained by some sequence, say  $ffgfg\cdots$ . Rather than considering the transforms  $f$  and  $g$ , we could equivalently make the corresponding product of matrices  $C$  and  $D$ . These and the resulting product is a non-singular matrix with (integer) non-negative coefficients. As such it has a single dominant eigenvalue, the so-called Perron-Frobenius (see Chapter XIII of the book [4] by Gantmacher) eigenvalue and a corresponding eigenvector. So, if we iterate the transform  $g \circ f \circ g \circ f \circ f$ , or equivalently  $DCDCC$ , the image in  $\mathbb{P}_2$  of the cone corresponding to the square  $[0, 1]^2$  accumulates around the line containing this eigenvector. This line is indeed a point of  $\mathcal{C}$ ; it corresponds to the code  $ffgfgffgfgffgfg\cdots$  where we repeat the pattern  $ffgfg$ . The points we now obtain are cubic, since so is the eigenvalue as a root of a cubic polynomial, namely the characteristic polynomial (of  $DCDCC$  in our example). Here again, we can localize these points by choosing a proper prefix.

*Additional properties.* Here are three additional properties of  $\mathcal{C}$ .

**Lemma 2.5.** *If  $f(P)$  belongs to  $\mathcal{C}$ , then  $P$  belongs to  $\mathcal{C}$ .*

*Proof.* We may assume that  $f(P) \neq (\frac{1}{3}, \frac{1}{3})$ . Indeed, if  $f(P)$  is in  $\mathcal{C}$ , then  $f(P)$  is a limit point of a sequence  $f^{a_1}g^{a_1}f^{a_2}g^{a_2}\dots$ . As  $f(P) \neq (\frac{1}{3}, \frac{1}{3})$  and the image of  $[0, 1]^2$  by  $f$  and  $g$  only intersect on this point, we deduce that  $a_1 > 0$ . By injectivity, we see that  $P = f^{a_1-1}g^{a_1}f^{a_2}g^{a_2}\dots$ , completing the proof.  $\square$

**Lemma 2.6.** *The three areas that are (1) the points that are strictly above  $\mathcal{C}$ , (2) the points that are on  $\mathcal{C}$  and (3) the points that are strictly below  $\mathcal{C}$  are stable under the action of  $f$  and  $g$ .*

*Proof.* Indeed  $f$  and  $g$  are injective maps. Let  $P = (u, v)$  be above  $\mathcal{C}$ . This means that the segment  $[P, P_0]$  where  $P_0 = (u, 1)$  does not cross  $\mathcal{C}$ . The segment  $[(0, 1), (1, 1)]$  on which  $P_0$  lies remains above our curve by construction after applying  $f$  or  $g$ . The segment  $[f(P), f(P_0)]$  may not cross  $\mathcal{C}$ , as a crossing point would be a  $f(Q)$ , and by Lemma 2.5,  $Q$  would belong to  $\mathcal{C}$  and to  $[P, P_0]$ , a contradiction.  $\square$

**Lemma 2.7.** *If a point  $P$  is below (resp. up of) the curve  $\mathcal{C}$ , then  $f(P)$  and  $g(P)$  are also there.*

*A simple continuous bound.* We now present a readily exploited upper bound for  $\mathcal{C}$ .

**Lemma 2.8.** *The graph of the function  $\theta_0 : x \mapsto 1 - x \log(1 + 1/x) / \log 2$  is stable under  $f$  and remains above  $\mathcal{C}$ . It crosses  $\mathcal{C}$  in three points:  $(0, 1)$ ,  $(\frac{1}{3}, \frac{1}{3})$  and  $(1, 0)$ .*

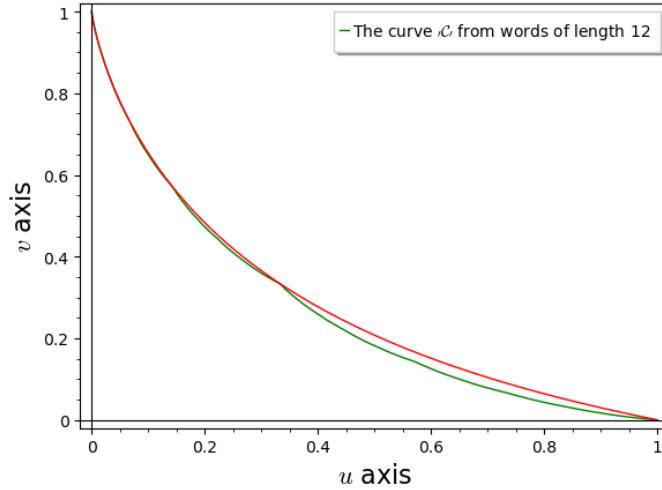


FIGURE 4. The curve  $\mathcal{C}$  in green (below) and the graph of  $\theta_0$  in red (above), drawing obtained through `ExpPairs.sage/plotC(12, 6) [1]+plotUpper(6)`.

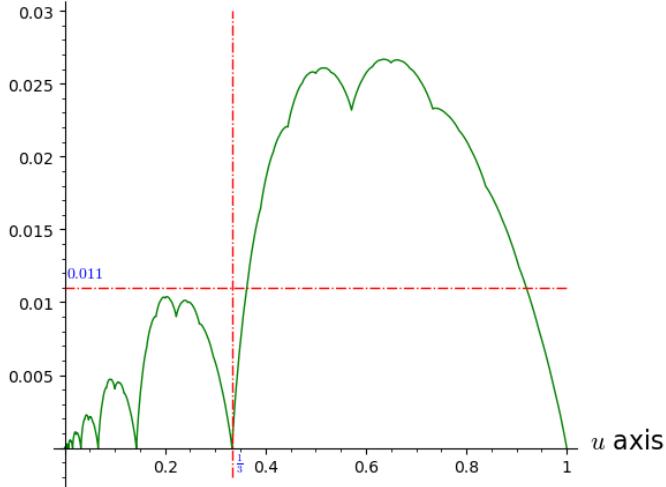


FIGURE 5. Difference between  $\theta_0$  and  $\mathcal{C}$ , drawing obtained by using `ExpPairs.sage/compareModelCurve(12)`.

*Proof.* Indeed, let us consider the region  $R = \{(u, v), v \leq \theta_0(u)\}$ . If a point  $P$  belongs to  $R$ , then  $f(P)$  also belongs to this region. Indeed, we compute that

$$\begin{aligned} \frac{v+1}{u+2} - \left(1 - \frac{u}{(u+2)\log 2} \log \frac{2u+2}{u}\right) \\ = \frac{1}{(u+2)\log 2} \left((v-1)\log 2 + u \log \frac{u+1}{u}\right) \\ \leq \frac{1}{(u+2)\log 2} \left(-u \log \frac{u+1}{u} + u \log \frac{u+1}{u}\right) = 0. \end{aligned}$$

This proof also shows that the graph of  $\theta_0$  is invariant under  $f$ . It is easy to show that this graph is above  $\mathcal{C}$  when  $u \in [1/2, 1]$ . The other parts of this graph are obtained by applying  $f$ , and since the region upper to  $\mathcal{C}$  is stable under  $f$ , the graph of  $\theta_0$  remains there.  $\square$

**Remark 2.9.** When  $P$  belongs to this region then  $g(P)$  also belongs to it. Indeed, we have

$$\begin{aligned} \frac{u}{u+2} - \left(1 - \frac{v+1}{(u+2)\log 2} \log \frac{v+u+3}{v+1}\right) \\ = \frac{1}{(u+2)\log 2} \left(-2 + (v+1) \log \frac{v+u+3}{v+1}\right). \end{aligned}$$

The function  $w \mapsto w \log(1 + U/w)$  is non-decreasing when  $w \in [1, 2]$  and  $U \in [2, 3]$  (its derivative is  $(\log z) + \frac{1}{z} - 1$  for  $z = (w+U)/w \in [2, 4]$ ). We thus only have to prove our assertion when  $v = \theta_0(u)$  and a simple plot is enough.

**Remark 2.10.** We note that upon choosing  $x = 1/(2^{k+1} - 1)$  for a positive integer  $k$ , we get the exponent pair

$$(x, \theta_0(x)) = \left(\frac{1}{2^{k+1} - 1}, \frac{2^{k+1} - k - 2}{2^{k+1} - 1}\right),$$

which is  $f^k(1, 0)$ , see page 60 of Graham-Kolesnik [6]. The actual definition of  $\theta_0$  was extrapolated from this formula.

*Proof of Theorem 1.2.* Translating Lemma 2.8 in terms of  $(\kappa, \lambda)$  and the definition of exponent pairs are all that is required to complete this proof. We note that  $\vartheta_0(\kappa) = (\theta_0(2\kappa) + 1)/2$   $\square$

### 3. CONVEX HULL

Given a symmetric subset  $S \subset [0, 1]^2$  that contains  $(0, 1)$ , we consider the smallest closed convex set  $\mathcal{C}(S)$  that contains all the images of  $S$  under  $f$  and  $g$ . Since our set contains  $(0, 1)$  it contains  $(1, 0)$  and  $(1/3, 1/3)$ . These are the vertices of the image of  $[0, 1]^2$  by  $f$  and  $g$ . On iterating, we find all the (opposite) vertices of the small parallelepipeds that we used to build  $\mathcal{C}$ , from which we conclude that the convex hull of  $\mathcal{C}$  is indeed in  $\mathcal{C}(\{(0, 1)\})$  which we denote by  $\mathcal{D}$ . Note that, since  $f$  and  $g$  preserve segments, it is enough to first iterate  $f$  and  $g$  and, in a second step, to take the convex hull of the final set.

As we see in Figure 2, the curve  $\mathcal{C}$  has a singular point at  $(\frac{1}{3}, \frac{1}{3})$ . But since any other location on the curve in a smooth image of the full curve, the set of points where this phenomenon occurs is in fact dense on  $\mathcal{C}$ .

Let us describe an algorithmical way of computing  $\mathcal{D}$ . We start from  $(0, 1)$  and  $(1, 0)$ , apply  $f$  and  $g$ , get the convex hull and repeat on the set of vertices obtained. Here is a plot of the first three steps.

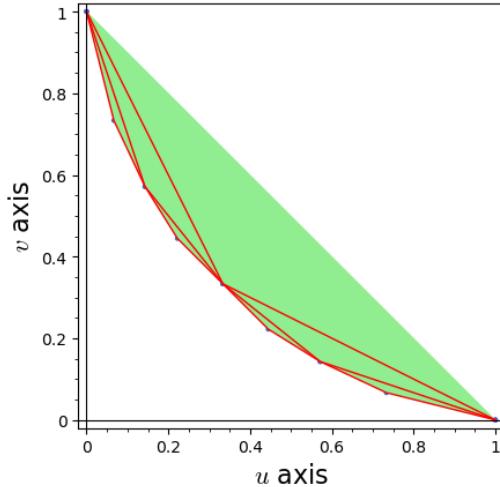


FIGURE 6. Approximate domain  $\mathcal{D}$  with words of length 3, drawing obtained via `ExpPairs.sage/plotDomainC(1)+plotDomainC(2)+plotDomainC(3)`.

At this level, the point  $(\frac{1}{3}, \frac{1}{3})$  is becoming useless, and in later steps, it will even become an interior point. So have reached the points

$(0, 1), (1/15, 11/15), (1/7, 4/7), (2/9, 4/9), (4/9, 2/9), (4/7, 1/7), (11/15, 1/15), (1, 0)$

Here is the situation when we reached the step 6, and which shows that finding a pattern to determine which points to keep and which to discard may be intricate.

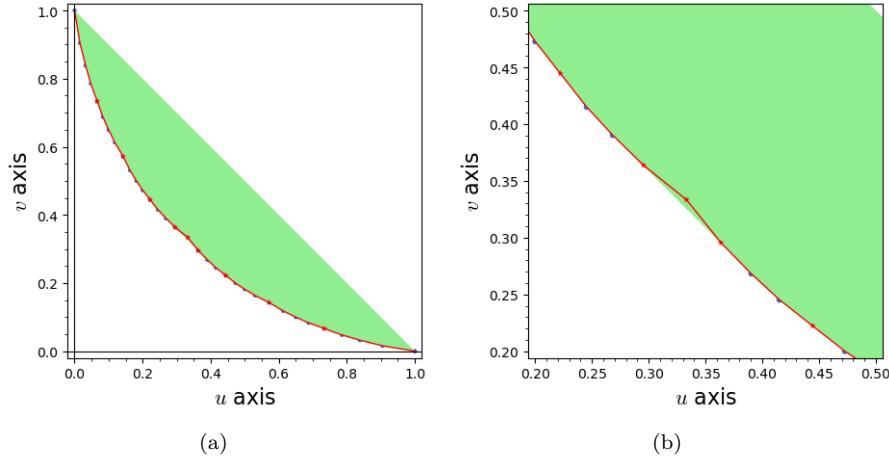


FIGURE 7. Approximate Domain  $\mathcal{D}$ . In blue round shape, the points of  $\mathcal{C}$  that are on the border, and in red star shape, the ones that are not needed anymore; zoom in (b) on the central part, drawing obtained via `ExpPairs.sage/plotDomainC(5, False)` and adding `.show(xmin=0.2, xmax=0.5, ymin=0.2, ymax=0.5, aspect_ratio=1)` for the second one.

#### 4. ADDING THE HUXLEY AND WATT POINT

In [1], E. Bombieri and H. Iwaniec improved the Lindelöf exponent beyond what was accessible through the exponent pair method. Their work was extended to yield an exponent pair by M.N. Huxley and N. Watt in [7] and by N. Watt in [13]. This gives us the two  $(u, v)$ -points

$$(10) \quad \left(\frac{9}{28}, \frac{9}{28}\right), \quad \left(\frac{89}{280}, \frac{89}{280}\right).$$

We should thus consider  $\mathcal{D}^* = \mathcal{C}(\{(0, 1), (\frac{89}{280}, \frac{89}{280})\})$ . We find that

$$f\left(\left(\frac{89}{280}, \frac{89}{280}\right)\right) = \left(\frac{89}{649}, \frac{369}{649}\right), \quad g\left(\left(\frac{89}{280}, \frac{89}{280}\right)\right) = \left(\frac{369}{649}, \frac{89}{649}\right).$$

Since the points given in (10) are below the curve  $\mathcal{C}$ , further points obtained by applying  $f$  and or  $g$  still stays there. Moreover, as we saw previously, a start like  $ffgfg$  localizes the image.

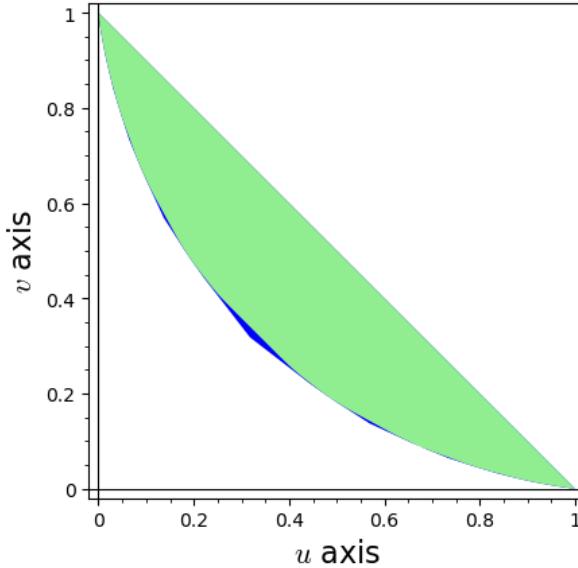


FIGURE 8. Domain  $\mathcal{D}^*$ . In blue, the part added by Watt's exponent pair, drawing obtained through `ExpPairs.sage/plotSimpleDomainCS([[0,1], [89/280,89/280]],12, 6, 'blue') + plotSimpleDomainCS([[0,1]], 12)`.

The next figure (in  $(\kappa, \lambda)$ ) shows that the adequation of our model  $\theta_0$  to this case is not as good as before but still within an acceptable margin.

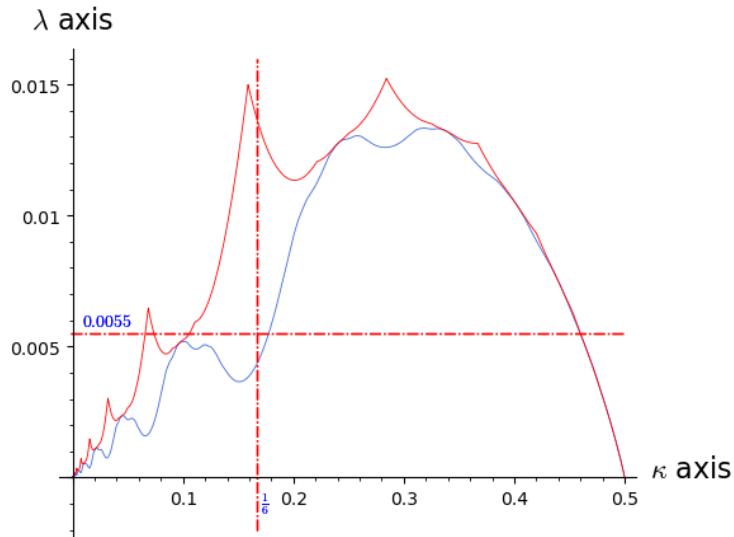


FIGURE 9. Difference between  $\theta_0$  and the optimal exponent pair from the Watt (resp. van der Corput) domain in red [top] (resp. in blue [below]), drawing obtained via `ExpPairs.sage/compareModelConvHull(8)+compareModelConvHull(8)`.

The question then is to find some equivalent form to Theorem 1.2. The function  $\theta_0$  (resp.  $\vartheta_0$  if we express it in the variables  $(\kappa, \lambda)$ ) has (rather strikingly) a simple form, but we may as well replace it by  $\theta^*$  (resp.  $\vartheta^*$ ) which parametrizes the (lower) border of  $\mathcal{D}^*$ .

**Theorem 4.1.** *The  $\theta^*$  be the continuous decreasing convex function on  $[0, 1]$  that parametrizes the lower border of  $\mathcal{D}^*$  and  $\vartheta^*$  be the continuous decreasing convex function on  $[0, 1/2]$  defined by  $\vartheta^*(\kappa) = (2\theta^*(2\kappa) + 1)/2$ . Let  $S$  be an exponential sum of monomial type and parameters  $N$  and  $F$ . Then, for every  $\kappa \in [0, \frac{1}{2}]$  and every  $\varepsilon > 0$ , we have  $S \ll_{\varepsilon} F^{\kappa+\varepsilon} N^{\vartheta^*(\kappa)+\varepsilon}$ . The functional equation  $\theta^* \circ \theta^* = \theta^*$  holds, as a consequence of the symmetry  $(u, v) \mapsto (v, u)$ .*

An upper bound for  $\theta^*$  is provided by the step-function  $\theta_1$  (or  $\vartheta_1$  in the variables  $(\kappa, \lambda)$ ) that links the following points:

$$\begin{aligned} [0, 1], & \left[ \frac{89}{1387}, \frac{1018}{1387} \right], \left[ \frac{89}{649}, \frac{369}{649} \right], \left[ \frac{369}{1667}, \frac{738}{1667} \right], \\ & \left[ \frac{89}{280}, \frac{89}{280} \right], \\ & \left[ \frac{738}{1667}, \frac{369}{1667} \right], \left[ \frac{369}{649}, \frac{89}{649} \right], \left[ \frac{1018}{1387}, \frac{89}{1387} \right], [1, 0]. \end{aligned}$$

This leads to a rather decent approximation of our border, as shown by the next plots.

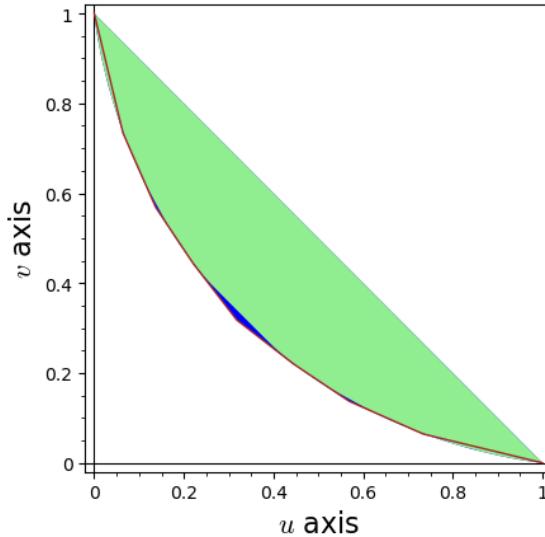


FIGURE 10. The Watt (resp. van der Corput) domain in blue (resp. in green) and the rational upper bound in brown.

This is obtained via

```
BasePlot = plotSimpleDomainCS([[0,1], [89/280,89/280]],10, 6, 'blue')
BasePlot += plotSimpleDomainCS([[0,1]], 10)
BasePlot += list_plot(rationalCHullW(2), zorder=3, color='brown',
                      plotjoined = True, figsize =6)
BasePlot.show()
```

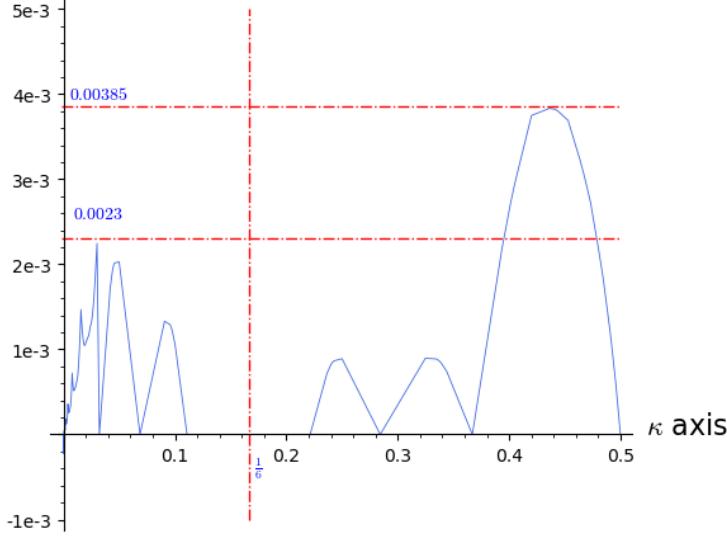


FIGURE 11. Difference  $\min(\vartheta_0, \vartheta_1)$  and the optimal exponent pair from the Watt domain,, drawing obtained via `ExpPairs.sage/compareFiniteModelConvHull(8)`

##### 5. A REMARK CONCERNING $p$ -DIMENSIONAL EXPONENT PAIRS

In [10], B.R. Srinivasan developed a multi-dimensional theory of exponent pairs. Roughly speaking, let  $\varphi(x_1, x_2, \dots, x_p)$  be a smooth 'polynomial-like' function whose partial derivative with respect to  $x_i$  remains of size  $F_i$  when  $x_i$  is about  $N_i$ . Then a pair  $(\kappa, \lambda) \in [0, \frac{1}{2(p+1)}] \times [\frac{2p-1}{2p}, 1]$  is said to be a  $p$ -dimensional exponent pair when, given the above data and  $\varepsilon > 0$ , we have

$$\sum_{\forall i, n_i \sim N_i} e^{2i\pi\varphi(n_1, \dots, n_p)} \ll_{\varepsilon} \left( \prod_i F_i \right)^{\kappa+\varepsilon} \left( \prod_i N_i \right)^{\lambda+\varepsilon}.$$

B.R. Srinivasan continues by showing that one can form two other  $p$ -dimensional exponent pairs from a given one, say  $(\kappa, \lambda)$ , by the expressions

$$\left( \frac{\kappa}{2(1+p\kappa)}, \frac{(2p-1)\kappa + \lambda + 1}{2(1+p\kappa)} \right), \quad \left( \lambda - \frac{1}{2}, \kappa + \frac{1}{2} \right).$$

The change of variables

$$u = 2p\kappa, \quad v = 2p\lambda - 2p + 1$$

leads to the rules (6), and the geometrical problem is thus unchanged! We then infer the next result from Theorem 1.2 and Theorem 4.1.

**Theorem 5.1.** *For any  $\kappa \in [0, \frac{1}{2(p+1)}]$ , the couple  $(\kappa, \frac{\vartheta_0(p\kappa)+p-1}{p})$  is a  $p$ -dimensional exponent pair. The same is true of  $(\kappa, \frac{\vartheta^*(p\kappa)+p-1}{p})$ , where  $\vartheta^*$  is defined in Theorem 4.1.*

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