LIMIT THEOREMS FOR CUSPIDAL ORBITS ON FUCHSIAN GROUPS

SANDRO BETTIN, SARY DRAPPEAU, AND JUNGWON LEE

Abstract. Let $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ be discrete, co-finite with cusps, and let $\mathfrak{b} \in \mathbb{R} \cup \{\infty\}$ be a cusp of Γ. Let $f : \Gamma\mathfrak{b} \to \mathbb{R}$ be a quantum modular form in the sense that the maps

$$\phi_\gamma(x) = f(x) - f(\gamma x) \qquad (x \in \Gamma\mathfrak{b}, \gamma \in \Gamma)$$

extend to Hölder-regular functions. Under bounded moments assumptions of the shape $\sum_n n^{-2} \int |\phi_{\gamma^n}(x)|^\alpha \, \mathrm{d}x < \infty$ for some $\alpha > 0$ and every fixed parabolic element $\gamma \in \Gamma$, we prove that the values $f(x)$ for $x \in \Gamma\mathfrak{b}$, ordered by growing denominators, tend to equidistribute according to a stable law. This generalizes to Fuchsian groups the work of Baladi-Vallée on rational orbits of the Gauss map.

In companion papers, we use this result, and in particular its flexible assumptions on ϕ_γ, to deduce limit theorems for automorphic L-series at the central point.

One ingredient in our proof is a particular cuspidal acceleration of the Bowen-Series map, whose inverse branches admit a convenient parametrization, and for which we describe explicitely a natural extension.

1. Introduction

Given an irrationnal number $x \in (0,1) \cap (\mathbb{R} \smallsetminus \mathbb{Q})$, the statistical behaviour of its continued fraction coefficients $(a_j(x))$ in

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}. \tag{1.1}$$

is an old theme in diophantine approximation. It is known that choosing x according to the absolutely continuous measure $\frac{\mathrm{d}x}{(1+x)\log 2}$, the coefficients $a_j(x)$ are distributed according to the Gauss-Kuzmin law on $\mathbb{N}_{>0}$, and that $(a_j(x), a_k(x))$ tend to be independently distributed as $|k-j| \to \infty$, to the effect that for sufficiently regular observables $\phi : \mathbb{N}_{>0} \to \mathbb{R}$, the partial sums

$$f_N(x) := \frac{1}{N} \sum_{n=0}^{N-1} \phi(a_j(x))$$

tend to be distributed, as $N \to \infty$, according to a Gaussian law with mean and variance of the order of N. This follows from the dynamical properties of the Gauss map, and more precisely the spectral properties of its transfer operator, see [Khi63, Bro96, AD01].

Instead of picking x irrational, choose $x \in (0,1) \cap \mathbb{Q}$ rational of denominator $\mathrm{den}(x) \leq Q$, independently at random among the $(\frac{3}{\pi^2} + o(1))Q^2$ such possibilities. The decomposition (1.1) is then finite with length $O(\log Q)$. It is natural to ask what remains of the statistical independence of $(a_j(x))_j$, or the limiting distribution of random variables of the shape

$$f(x) = \sum_j \phi(a_j(x)) \tag{1.2}$$

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given a fixed map $\phi : \mathbb{N} \to \mathbb{R}$.

This question was initially studied due its link with the complexity, in various aspects, of the Euclidean algorithm for division [Hen94]. It was fully resolved in the works of Vallée and Baladi-Vallée [Val03, BV05b], who relate the problem to the spectra of a two-parameter family of twists of the Gauss-Kuzmin operator.

In [BD22, LS25, DN26], these ideas were pushed further, motivated by questions of a different kind, namely the value distribution of trigonometric series formed with coefficients of modular forms,

$$f(x) = \sum_{n\geq 1} \frac{a_n}{\sqrt{n}} \mathrm{e}^{2\pi i n x}, \qquad (x \in \mathbb{Q}) \tag{1.3}$$

where a_n are Fourier coefficients of an automorphic form for a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$. The main observation which relates the two questions is the fact that the map

$$h(x) := f(x) - f(-1/x) \tag{1.4}$$

initially defined on $\mathbb{Q} \smallsetminus \{0\}$, extends to a continuous map on $\mathbb{R}$. The relation (1.4), along with the periodicity of f, shows that $f(x)$ is essentially of the shape (1.2) with $\phi(a_j(x))$ replaced by $h(T^j x)$, where $T(x) := \{1/x\}$ is the Gauss map. The 1-periodic maps for which the period function defined in (1.4) satisfies some regularity property (in a purposedly vague sense) are referred to as quantum modular forms in Zagier's seminal work [Zag10].

Series of the shape (1.3) exist for automorphic forms on Fuchsian groups Γ other than $\mathrm{SL}(2,\mathbb{Z})$, and motivated by specific examples we are led wonder about the statistical distribution of those generalizations (Theorem 1.5 and Corollaries 1.6 and 1.7 below). It is natural to try to extend the proof of [BD22, LS25] to this setting. The main obstacle comes from the dynamical systems arguments used in the proof, which originate from Baladi-Vallée's work [BV05b] on the Gauss map. The purpose of the present paper is to work out a generalization of Baladi-Vallée's work, with flexible hypotheses on the observables, which can be used to understand series of the shape (1.3) for automorphic forms on more general Fuchsian groups.

Let $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ be a discrete and co-finite subgroup. Denote $C(\Gamma) \subset \mathbb{P}^1(\mathbb{R})$ the set of cusps of Γ (the fixed points of parabolic matrices), and assume that ∞ is a cusp of Γ. The set of cusps is equiped with a notion of denominator, denoted $\mathrm{den}(x)$, which will be recalled below, and which coincides with the denominator of numbers in $C(\Gamma) = \mathbb{Q}$ in the case $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. Given a finite set $S \subset C(\Gamma)$, a map $f : C(\Gamma)\backslash S \to \mathbb{C}$ is called a quantum modular form (QMF) of weight 0 for Γ if for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, the "error to modularity" function

$$h_\gamma(x) := f(x) - f(\gamma x), \quad x \in C(\Gamma)\backslash(S \cup \gamma^{-1} S)$$

satisfies certain regularity property (in this introduction we focus on the trivial multiplier system).

The main result of the present paper can be described informally as follows:

Suppose that for some parameters $\alpha, \kappa_0, \lambda > 0$, the maps (h_γ) extend to κ_0-Hölder continuous maps on $\mathbb{R} \smallsetminus \{\gamma^{-1}\infty\}$, in such a way that for every fixed parabolic matrix $\gamma \in \Gamma$ we have

$$\sum_{n\geq 1} \frac{1}{n^2} (\|h_{\gamma^n}\|_\infty^\alpha + \|h_{\gamma^n}\|_{(\kappa_0)}^\lambda) < \infty$$

where $\|\cdot\|_{(\kappa_0)}$ *denotes the Hölder semi-norm. Then as* x *varies among elements of* $C(\Gamma)$ *of denominators at most* Q*, and* $Q \to \infty$*, the values* $f(x)$ *tend to equidistribute according to a stable law, whose parameters are determined from the asymptotic behaviour of* $h_{\gamma^n}(x)$ *in certain ranges in* n *and* x*.*

We now introduce precisely our notations. The precise statement of our main theorem will be presented in Section 1.7, with an overview of its proof given in Section 1.9.

1.1. **Disk model and assumptions on the group.** First we switch to the disk model for the hyperbolic surface, which will be more convenient for our later arguments. Let $\mathbb{D}$ be the open unit disc, and

$$p_{\mathbb{D}} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}, \qquad p_{\mathbb{D}}(z) := \frac{z-i}{z+i}.$$

This maps conformally the set $\mathbb{H}$ to $\mathbb{D}$. We also denote $p_{\mathbb{D}}$ the element of $\mathrm{SU}(1,1)$ which corresponds to the homography $p_{\mathbb{D}}$.

We consider a discrete subgroup $G \subset \mathrm{SU}(1,1) = \left\{ \left(\begin{smallmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{smallmatrix}\right), |\alpha|^2 - |\beta|^2 = 1 \right\}$. We assume that G satisfies the following hypotheses:

(a) G is finitely generated.
(b) The surface $G\backslash\mathbb{D}$ has finite volume.
(c) There is a cusp at 1.
(d) There is no elliptic element in G.
(e) $-\,\mathrm{id} \notin G$.

The main assumptions are (a)-(c). The main point of condition (c) is to ensure that we have at least one cusp, with respect to which the denominators of other cusps will be defined. The fact that the cusp is at 1 simplifies the presentation of some of our arguments, and can be assumed without loss of generality by conjugation. In all applications we consider, the remaining assumptions (d) and (e) will be implemented without loss of generality by restricting to a suitable subgroup; they are used to simplify some aspects of our arguments.

1.2. **Orbits, denominators and a probability space.** Let $C(G)$ be the set of cusps of G on $\partial\mathbb{D}$. Consider a cusp $\mathfrak{b} \in C(G)$, with an associated scaling matrix $s_{\mathfrak{b}} \in \mathrm{SU}(1,1)$, meaning that the stabilizer $G_{\mathfrak{b}}$ of $\mathfrak{b}$ in G satisfies

$$\mathfrak{b} = s_{\mathfrak{b}} 1, \qquad G_{\mathfrak{b}} = s_{\mathfrak{b}} \left\{ \left(\begin{smallmatrix} 1+ki/2 & -ki/2 \\ ki/2 & 1-ki/2 \end{smallmatrix}\right), k \in \mathbb{Z} \right\} s_{\mathfrak{b}}^{-1}.$$

Given $x \in G\mathfrak{b}$, we define a map by setting

$$\Upsilon(x) = \Upsilon_{\mathfrak{b}}^{G}(x) := \overline{\alpha + \beta}, \qquad \left(x = g\mathfrak{b}, \quad g \in G, \quad g s_{\mathfrak{b}} = \left(\begin{smallmatrix} \alpha & \beta \\ * & * \end{smallmatrix}\right)\right). \tag{1.5}$$

From now on, we will keep G and $\mathfrak{b}$ fixed and drop them from the notation.

A routine computation shows that $\overline{\alpha + \beta}$ depends only the class $g \in G/G_{\mathfrak{b}}$, which justifies the notation $\Upsilon(x)$. Moreover, the imaginary part

$$q(x) := \mathfrak{Im}\,\Upsilon(x) \tag{1.6}$$

depends only on the class x modulo left-translation by G_1.

Next, we let

$$\psi : \partial\mathbb{D} \to \mathbb{R}_+$$

be a smooth and not identically zero test function. We assume that $1 \notin \operatorname{supp}\psi$. We define a set, depending on $\mathfrak{b}$, by

$$\Omega_Q := \{x \in G\mathfrak{b}, |q(x)| \leq Q\}, \qquad (Q \geq 2).$$

For $g \in G$, write $x = g\mathfrak{b} = \frac{\alpha+\beta}{\bar{\alpha}+\bar{\beta}}$ with $gs_{\mathfrak{b}} = \left(\begin{smallmatrix} \alpha & \beta \\ * & * \end{smallmatrix}\right)$. Translating back to the upper-half plane model $gs_{\mathfrak{b}} = p_{\mathbb{D}} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) p_{\mathbb{D}}^{-1}$, we get

$$\alpha + \beta = a - ic \tag{1.7}$$

and therefore $q(x) = c$. The set $G_1 \backslash \Omega_Q$ thus corresponds, in the disk model, to the set $T_{\infty\mathfrak{b}}(Q)$ defined in Sections 2.2 and 3.3 of [PR18] up to scaling at ∞: more precisely, denoting $\Gamma_\infty := p_{\mathbb{D}}^{-1} G_1 p_{\mathbb{D}} = \left(\begin{smallmatrix} 1 & \omega\mathbb{Z} \\ 0 & 1 \end{smallmatrix}\right)$ with $\omega > 0$, we check that[1]

$$G_1 \backslash \Omega_Q = \omega T_{\infty,\mathfrak{b}}(Q\sqrt{\omega}). \tag{1.8}$$

By [PR18, Theorem 3.8], the set $G_1 \backslash \Omega_Q$ is finite. On the other hand, since ψ vanishes on a neighborhood of 1, we have

$$\sup_{x \in \partial\mathbb{D}} \operatorname{card}\{\tau \in G_1, \psi(\tau x) \neq 0\} < \infty.$$

Summing over a set of representatives of $G_1 \backslash \Omega_Q$, we deduce that

$$\operatorname{card}\{x \in \Omega_Q : \psi(x) \neq 0\} < \infty.$$

Using a result from [PR18], we estimate the total mass of ψ on Ω_Q by

$$\sum_{x \in \Omega_Q} \psi(x) \sim c_\psi Q^2 \tag{1.9}$$

as $Q \to \infty$, for some explicit constant $c_\psi > 0$ depending only on ψ and G. This will be made precise in Lemma 2.2 below.

In view of the above, we may define, for Q large enough, a probability measure on Ω_Q proportional to ψ. We denote by $\mathbb{P}_Q$ and $\mathbb{E}_Q$ the associated probability and expectation, so that for any map f on $C(G)$, we have

$$\mathbb{E}_Q[f(x)] = \frac{\sum_{x \in \Omega_Q} \psi(x) f(x)}{\sum_{x \in \Omega_Q} \psi(x)}.$$

Remark 1.1. In [BV05b, BD22, LS25], the average is instead performed over $\Omega_Q \cap I$ where I is a fundamental interval for the action of G_1, *e.g.* the unit interval in the upper-half plane model. We can recover this case by taking ψ such that $\sum_{\tau \in G_1} \psi(\tau x) = 1$ (a smooth partition of unity), since Ω_Q is invariant under left translation by G_1.

1.3. **Fundamental domain.** Let R be a fundamental domain in $\mathbb{D}$ under the action of G, whose sides are non-intersecting infinite geodesics. The fact that such a domain R exists is proved in [Tuk73, Proposition p. 15] under the hypotheses (a)-(d). Since R has finite volume, by Siegel's Theorem [Kat92, Theorem 4.1.1], the closure of R in $\overline{\mathbb{D}} = \mathbb{D} \cup \partial\mathbb{D}$ intersects $\partial\mathbb{D}$ at a finite set V, and therefore R is an ideal polygon.

A side s of R is paired with a unique side of R, say $\widehat{s}$, by an element of $g_s \in G \smallsetminus \{1\}$, in such a way that

$$s = g_s \widehat{s}.$$

We gather in Section 2.2 below some useful facts about this pairing. In particular, the map $s \mapsto \widehat{s}$ is an involution with no fixed point. Given s, let $I(s) \subset \partial\mathbb{D}$ be the closed interval whose ends are the vertices delimiting s, and which does not intersect other vertices of R. The intervals $\{I(s)\}$ then form a partition of $\partial\mathbb{D}$ up to vertices.

[1]Note that $T_{\infty,\mathfrak{b}}$ in the right-hand side depends on a choice of scaling matrix σ_∞ at ∞, although the left-hand side does not; the relation (1.8) as stated holds with the choice $\sigma_\infty = \left(\begin{smallmatrix} \sqrt{\omega} & 0 \\ 0 & 1/\sqrt{\omega} \end{smallmatrix}\right)$.

For each side s, we define the subset $L_s \subset \partial\mathbb{D}$ of the circle, consisting of all the arcs which do not touch $I(\widehat{s})$:

$$L_s := \bigcup_{s' : I(s') \cap I(\widehat{s}) = \varnothing} I(s').$$

Under the action of g_s, the set L_s is mapped into the interior of $I(s)$. We let

$$K_s := g_s(L_s), \qquad K := \bigcup_{s\text{: side}} K_s.$$

Each K_s is an interval, bounded away from the vertices of R, and the set (K_s) are disjoint. In Figure 1 on page 13, we draw a fundamental domain for the principal congruence subgroup $\Gamma(3)$.

1.4. **Multiplier system.** Fix an integer $d \geq 1$ and let $\Theta \subset \mathrm{GL}(d, \mathbb{R})$ be a finite subgroup. We also let

$$\partial\mathbb{D}' := \partial\mathbb{D} \setminus V. \tag{1.10}$$

We consider a family of maps indexed by the elements of G,

$$(\vartheta_g)_{g \in G}, \qquad \vartheta_g : \partial\mathbb{D}' \to \Theta, \tag{1.11}$$

satisfying the cocycle property, or derivative rule:

$$\vartheta_{g_1 g_2}(x) = \vartheta_{g_2}(x)\vartheta_{g_1}(g_2 x), \qquad (x,\ g_2 x \notin V). \tag{1.12}$$

We assume two properties on the map ϑ_g, which are informally:

- The map ϑ_g being constant on each arc $I(s)$, for each $g \in G$;
- A form of strong mixing, in the sense that any value $\vartheta_0 \in \Theta$ is in the range of ϑ_g with some constraints on the shape of g.

The precise assumptions will be described in Section 5.1 below.

1.5. **Functional space.** The following norms will be relevant for our later arguments.

Definition 1.2 (Piecewise κ-Hölder space)**.** *Given $\kappa \in (0, 1]$, let H^κ be the set of bounded maps $f : K \times \Theta \to \mathbb{R}^d$ which have a finite κ-Hölder semi-norm*

$$\|f\|_{(\kappa)} := \sup_{\substack{s\text{: side} \\ x,y \in K_s \\ \vartheta_0 \in \Theta}} \frac{|f(x, \vartheta_0) - f(y, \vartheta_0)|}{|x - y|^\kappa} < \infty.$$

See also [Eva10, section 5.1]. We equip H^κ with two norms:

$$\|f\|_\infty := \sup_{(x,\vartheta_0) \in K} |f(x, \vartheta_0)|, \qquad \|f\|_\kappa := \|f\|_\infty + \|f\|_{(\kappa)}.$$

With slight abuse of notation, we use the same notations for H^κ and its norms also for functions on K, extending them trivially to $K \times \Theta$.

1.6. **Quantum modular forms.** Consider a family of maps from K to $\mathbb{R}^d$,

$$(\phi_g)_{g \in G} \subset H^{\kappa_0}$$

for some $\kappa_0 > 0$. We are interested in maps

$$\mathfrak{f} : C(G) \to \mathbb{R}^d$$

which are quantum modular forms in the following sense.

Hypothesis 1. *For all $g \in G$, we have*

$$\vartheta_g(x)\mathfrak{f}(gx) = \mathfrak{f}(x) + \boldsymbol{\phi}_g(x), \qquad (x \in C(G) \cap K, g \in G). \tag{1.13}$$

Moreover,

(a) the multiplier (ϑ_g) satisfies the stability and mixing hypotheses described in Section 5.1 below;
(b) there exist $\kappa_0, \alpha, \lambda > 0$ such that the period functions $(\boldsymbol{\phi}_g)$ belong to H^{κ_0} and have moments and Hölder-norm moments of order α and λ respectively, where these moments are defined in Section 6 below.

We note that this hypothesis as stated is stronger than needed: the actual hypothesis concerns only a subsequence of $(\boldsymbol{\phi}_g)$ along specific matrices related to K; we defer to Sections 5.5 and 6 for the precise hypothesis and set-up.

Hypothesis 1 implies that the map $(g, x) \mapsto \boldsymbol{\phi}_g(x)$ is a cocycle in the sense that

$$\boldsymbol{\phi}_{g_1 g_2}(x) = \boldsymbol{\phi}_{g_2}(x) + \vartheta_{g_2}(x)\boldsymbol{\phi}_{g_1}(g_2 x) \tag{1.14}$$

for all $g_1, g_2 \in G$, $x, g_2 x \in K$.

1.7. **Main statement.** Our main result is an asymptotic estimate for the characteristic function of $\mathfrak{f}(x)$, viewed as a random variable on Ω_Q, as $Q \to \infty$.

For a vector $\boldsymbol{t} \in \mathbb{R}^d$, we let $|\boldsymbol{t}|$ be the Euclidean norm of $\boldsymbol{t}$ and denote by $\langle \cdot, \cdot \rangle$ the corresponding scalar product.

Theorem 1.3. *Assume $\mathfrak{f}$ satisfies Hypothesis 1 with parameters $\kappa_0, \alpha, \lambda$. Then there exist $t_0, \delta_0 > 0$, and two maps*

$$U_1, U_2 : [-t_0, t_0]^d \to \mathbb{C},$$

such that for all $\boldsymbol{t} \in \mathbb{R}^d$ with $|\boldsymbol{t}| < t_0$, we have

$$\mathbb{E}_Q(\mathrm{e}^{i\langle \boldsymbol{t}, \mathfrak{f}(x)\rangle}) = \exp\Big(U_1(\boldsymbol{t}) \log Q + U_2(\boldsymbol{t}) + O(Q^{-\delta_0})\Big),$$

where the implicit constant depends only on κ_0 and the values of the moments in Hypothesis 1.

Additionally, there is a constant $\mathfrak{d} > 0$ independent from $\mathfrak{f}$ for which the following holds. There exists a random variable $\Phi \in \mathbb{R}^d$ such that for all $\varepsilon > 0$, we have

$$\mathbb{E}(|\Phi|^{\alpha-\varepsilon}) < \infty,$$

and the following holds:

– *The maps U_1, U_2 satisfy*

$$U_1(\boldsymbol{t}) = \mathfrak{d}^{-1}\mathbb{E}(\mathrm{e}^{i\langle \boldsymbol{t}, \Phi\rangle} - 1) + O_\varepsilon(|\boldsymbol{t}|^2 + |\boldsymbol{t}|^{2\alpha-2\varepsilon}),$$
$$U_2(\boldsymbol{t}) = O_\varepsilon(|\boldsymbol{t}| + |\boldsymbol{t}|^{\alpha-\varepsilon}).$$

– *Moreover if $\alpha > 2$, then there exists $\mu_\Phi \in \mathbb{R}^d$ and a real $d \times d$ positive semi-definite matrix Σ_Φ such that*

$$U_1(\boldsymbol{t}) = i\langle \boldsymbol{t}, \mu_\Phi\rangle - \tfrac{1}{2}\boldsymbol{t}^T \Sigma_\Phi \boldsymbol{t} + O_\varepsilon(|\boldsymbol{t}|^3 + |\boldsymbol{t}|^{1+\alpha-\varepsilon}).$$

We have $\mu_\Phi = \mathfrak{d}^{-1}\mathbb{E}(\Phi)$.

If in Hypothesis 1, condition (b) is satisfied with $\alpha > 2$, then Theorem 1.3 implies, via Lévy's continuity theorem, convergence to a non-degenerate Gaussian law of the ratio

$$\frac{\mathfrak{f}(x) - \mu_\Phi \log Q}{\sqrt{\log Q}}$$

as long as $\boldsymbol{\phi}_g(x)$ is not essentially $\log|\frac{d}{dx}gx|$ for all $g \in G$ (cf. [BD22, Corollary 3.2]).

We will give, in the course of our arguments, explicit formulas for Φ. However, the expression that we obtain depends on a non-canonical choice of induction process.

Remark 1.4. In [BV05b, Section 5], Baladi and Vallée also prove a local limit theorem for observables which are lattice-values. Some of their arguments could be adapted to our setting, however the need to avoid observables proportional to log, and the possible combined behaviour in several dimensions, makes it prohibitive to give a simple representative statement. Since we do not have in mind presently any particular arithmetic application which could motivate such a statement, we refrain to carry this out here.

1.8. **Application to trigonometric series with $GL(2)$ automorphic coefficients.** We briefly mention the applications which motivated Theorem 1.3. The details will appear in [BDLa] and [BDLb].

Our main motivating application is to the limiting distribution of the values

$$L(x) \overset{AC}{:=} \sum_{n=1}^{\infty} \frac{a(n)}{\sqrt{n}} \mathrm{e}^{2\pi i n x} \tag{1.15}$$

where $a(n)$ are the Fourier coefficients of a non-zero automorphic form ϕ of integer weight for a cofinite subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ with a cusp at ∞, and x varies among cusps of Γ. For all such x, the series (1.15) is defined by means of analytic continuation of the series $\sum_n a(n)\mathrm{e}^{2\pi i n x}/n^s$ initially defined for large $\mathfrak{Re}(s)$; we write AC above the $:=$ sign to indicate this. An important point is that this analytic continuation is highly sensitive to the properties of x as a cusp of Γ. We normalize the Fourier coefficients $a(n)$ so that the Rankin-Selberg theorem implies $\sum_{n\le x} |a(n)|^2 \asymp x$, and we let λ_ϕ denote the Laplace eigenvalue of ϕ.

Theorem 1.5. *Suppose that either ϕ is cuspidal, or $\lambda_\phi \ge 1/4$. Then there exists $\sigma_\phi > 0$ and $\beta_\phi \in \{0, 1, 3\}$ such that, as x varies among cusps of Γ of denominators bounded by Q and $Q \to \infty$, the values*

$$\frac{L(x)}{\sigma_\phi (\log Q)^{1/2} (\log\log Q)^{\beta_\phi}}$$

tend to equidistribute according to a standard complex Gaussian density.

This statement generalizes various earlier results notably by Petridis-Risager [PR18], Nordentoft [Nor21], Lee-Sun [LS25], Bettin-Drappeau [BD22] and [DN26]. The value of β_ϕ can be determined from the growth of ϕ at cusps, and we have $\beta_\phi = 0$ if ϕ is cuspidal. Moreover, when ϕ is cuspidal, we can obtain an explicit expression for σ_ϕ in terms of ϕ. We refer to [BDLa] for more details.

Two new examples of sequences $a(n)$, defined in elementary ways, for which the present work applies, are the following.

Corollary 1.6. *The statement of Theorem 1.5 holds true, with $\beta_\phi = 1$, for the series*

$$L(x) \overset{AC}{=} \sum_{n=1}^{\infty} \frac{r_2(n)}{\sqrt{n}} \mathrm{e}^{2\pi i n x}$$

where $r_2(n)$ is the number of ways that n can be written as sums of two squares.

Corollary 1.7. *The statement of Theorem 1.5 holds true, with $\beta_\phi = 0$ and $\sigma_\phi = 1/\sqrt{2}$, for the quadratic Weyl sum*

$$L(x) \stackrel{AC}{=} \sum_{n=1}^{\infty} \frac{\mathrm{e}^{2\pi i n^2 x}}{\sqrt{n}}.$$

The corresponding automorphic forms are an Eisenstein series of weight 1 for $\Gamma_0(4)$, and a Theta series of half-integer weight for the same group, respectively. In corollaries 1.6 and 1.7, the parameter x varies among rational numbers in $[0,1]$ of usual denominator at most Q. We refer to [BDLa] for more details and references.

In forthcoming work [BDLb], we will study applications of the present methods to partial Fourier series with automorphic coefficients.

1.9. **Overview.** Theorem 1.3 is a wide extension of Theorem 3.1 of [BD22] and Theorem A [LS25], which in turn generalized work of Baladi and Vallée [BV05b]. We briefly explain the new features in the proof.

The first question is to find a proper analogue of the Gauss map: a Markov, hyperbolic map on $\partial\mathbb{D}$, acting locally as homographies associated with elements of G. This problem, stated precisely in these terms, was resolved by Bowen–Series [BS79] using a geometric construction based on a fundamental domain for G. An induction of the Bowen-Series map, which we denote $\mathcal{F}_K$, then possesses spectral properties analogous to those of the Gauss map.

Generalizing the arguments of Baladi-Vallée [BV05b] to this setting requires solving several issues. We properly parametrize the coding and domains of inverse branches of the induced map $\mathcal{F}_K$; along the way, we take the opportunity to give an explicit expression for the natural extension of $\mathcal{F}_K$ and its invariant measure, by exploiting the flexibility we have of choosing a convenient fundamental domain, namely an ideal hyperbolic polygon. Then we are to provide precise estimates for the size of inverse branches. Once this is at hand, the rest of the arguments of Baladi-Vallée, and in particularly the Dolgopyat method (with the strong UNI property), could be generalized without major issue, if one is ready to accept strong assumptions on the "observables" $(\boldsymbol{\phi}_g)$.

One of our main challenges, however, lies in the fact that for our applications in [BDLa], the maps $(\boldsymbol{\phi}_g)$ are difficult to analyze, for lack of a simple explicit expression. In particular, we are to avoid a hypothesis relying on estimates uniform in g varying among the inverse branches of $\mathcal{F}_K$. This issue is addressed by considering a new induction $\mathcal{F}_c$ of the Bowen-Series map, whose inverse branches, along with their domains, are very easy to characterize: inverse branches of $\mathcal{F}_c$ are parametrized by homographies essentially of the shape $g^n x$, for $n \geq 1$ arbitrary but g varying over a *finite* set of parabolic elements of G. We define this map in Section 3.2 below, and make a precise study of its natural extension and its invariant measure. The original Bowen-Series accelerated map $\mathcal{F}_K$ is an induction of the map $\mathcal{F}_c$.

The map $\mathcal{F}_c$ is closely related to the cuspidal acceleration introduced by Artigiani, Marchese and Ulcigrai in [AMU16] (see also [Mar18]), which we denote $\mathcal{F}_{\mathrm{AMU}}$ here. The differences between the two settings are discussed precisely in Section 3.3 below. We have taken the opportunity here to work out, additionally, the natural extension and the invariant measure for $\mathcal{F}_{\mathrm{AMU}}$.

The map $\mathcal{F}_c$ is used in our arguments on two occasions: firstly, as a fundamental tool to establish analytic continuation of a certain generating series (Section 5.3); and secondly, as a way to effectively describe the random variable Φ in Theorem 1.3 (Section 8.2).

Organisation of the paper. We outline the contents as follows.

– In Section 2 and 3, we define the instance of the Bowen–Series map we will use. We then introduce and discuss several accelerations of the Bowen-Series map away from vertices of the fundamental domains, in particular the maps $\mathcal{F}_K$ and $\mathcal{F}_c$ alluded to in the above overview. We explicitely describe their inverse branches, and construct associated natural extensions and invariant densities.
– In Section 4, we recall the main metric properties on inverse branches of the accelerated maps. This includes contraction estimates, and also estimates relative to the verification of the uniform non-integrability conditions (UNI).
– In Section 5 we introduce the various twists of the Ruelle-Perron-Frobenius operators $\mathscr{L}$ and $\mathscr{C}$ associated with $\mathcal{F}_K$ and $\mathcal{F}_c$, and relate $\mathscr{C}$ with $\mathscr{L}$.
– In Section 6, we define rigorously the sup-norm and Hölder-norm moments bounds on $(\boldsymbol{\phi}_g)_g$ which constitute Hypothesis 1.(b).
– In Section 7, we show a contraction estimate for large twists of the transfer operator $\mathscr{L}$, following the arguments in [Dol98, BV05b, BD22, LS25], and in Section 8 we analyse the asymptotic behaviour of the leading eigenvalue.
– Finally, in Sections 9 and 10, we define the generating series relevant for our problem and prove its meromorphic continuation to the left of $\mathfrak{Re}(s) = 1$, and in Section 10.1, we deduce the main theorem 1.3.

Notations. We use the notation $\mathbf{1}_A$ for the indicator of a set or a logical condition. We denote $f = O(g)$, or equivalently $f \ll g$, to say that there is a constant $C > 0$ for which $|f| \leq Cg$. The domain of validity will be clear from the context, and the implicit constant C may depend on the group G or the choice of fundamental domain R. In cases where C depends on other paremeters in the analysis, this dependence will be explicited in subscript, such as $f \ll_* g$ or $f = O_*(g)$, or else will be described explicitely. For all $z \in \mathbb{C}$, we define $\mathrm{e}(z) = \mathrm{e}^{2\pi i z}$.

We let $\mathrm{spr}(\mathscr{T})$ denote the spectral radius of an operator $\mathscr{T}$. We let $\|f\|_\infty$ we indicate the usual sup-norm on the domain of f. Given a norm or seminorm $\|\cdot\|_*$ on a functional space, we will write $\|\cdot\|_{*,D}$ the same norm on the subspace of functions restricted to D. We use $|\cdot|$ to indicate the usual Euclidean norm on $\mathbb{R}^d$.

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2. Bowen–Series map

2.1. **Denominators and orbits.** Recall that the quantity $\Upsilon(x)$ was defined in (1.5), for $x \in G\mathfrak{b}$. Given $g = \left(\begin{smallmatrix} * & * \\ u & v \end{smallmatrix}\right)$, we define

$$j(g, x) = ux + v.$$

For all $g_1, g_2 \in G$ and $x \in \overline{\mathbb{D}}$, we have

$$j(g_1 g_2, x) = j(g_1, g_2 x) j(g_2, x),$$

see [Iwa97, eq. (2.3)]. It is then easily seen that

$$\Upsilon(gx) = j(g, x)\Upsilon(x), \qquad (x \in G\mathfrak{b}, g \in G). \tag{2.1}$$

Lemma 2.1. *For all $\sigma > 1$, we have*

$$\sum_{x \in G\mathfrak{b}} |\Upsilon(x)|^{-2\sigma} < \infty.$$

Proof. We translate the problem to the $\mathrm{SL}(2, \mathbb{R})$ setting, where the analogous estimate is essentially [Iwa02, Proposition 2.8]. We first assume that $\mathfrak{b} = 1$. We keep the notation $g = \left(\begin{smallmatrix} \alpha & \beta \\ * & * \end{smallmatrix}\right)$ throughout the computation, and given such g, define $\gamma = p_{\mathbb{D}}^{-1} g p_{\mathbb{D}} \in \mathrm{SL}(2, \mathbb{R})$, which we write $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Let $\Gamma = p_{\mathbb{D}}^{-1} G p_{\mathbb{D}}$, so that $\Gamma_\infty = p_{\mathbb{D}}^{-1} G_1 p_{\mathbb{D}}$.

By (1.7) our sum becomes

$$\begin{aligned} \sum_{g \in G/G_1} |\alpha + \beta|^{-2\sigma} = \sum_{\gamma \in \Gamma/\Gamma_\infty} |a - ic|^{-2\sigma} &= 1 + \sum_{\substack{\gamma \in \Gamma_\infty \backslash \Gamma/\Gamma_\infty \\ c \neq 0}} \sum_{n \in \omega\mathbb{Z}} \left|(a + cn)^2 + c^2\right|^{-\sigma} \\ &\ll 1 + \sum_{\substack{\gamma \in \Gamma_\infty \backslash \Gamma/\Gamma_\infty \\ c \neq 0}} |c|^{-2\sigma} \end{aligned}$$

where $\omega > 0$ is such that $\Gamma_\infty = \left(\begin{smallmatrix} 1 & \omega\mathbb{Z} \\ 0 & 1 \end{smallmatrix}\right)$. At this point we invoke [Iwa02, eq. (2.33)], to the effect that the sum over γ is bounded as long as $\sigma > 1$.

Assume now that $\mathfrak{b}$ is arbitrary and let $G' := s_{\mathfrak{b}}^{-1} G s_{\mathfrak{b}}$. By definition, 1 is a cusp of G', and by what we have just proved, we have

$$\sum_{x \in G'1} |\Upsilon(x)|^{-2\sigma} < \infty, \qquad (\sigma > 1). \tag{2.2}$$

Our original sum can be related this one by the following computation:

$$\sum_{x \in G\mathfrak{b}} |\Upsilon(x)|^{-2\sigma} = \sum_{x \in G'1} |\Upsilon(s_{\mathfrak{b}} x)|^{-2\sigma} = \sum_{x \in G'1} |\Upsilon(x)|^{-2\sigma} |j(s_{\mathfrak{b}}, x)|^{-2\sigma}.$$

Here, it is easily checked that the map $j(s_{\mathfrak{b}}, \cdot)$ is bounded and non-zero on the circle, so that $|j(s_{\mathfrak{b}}, x)| \asymp 1$. This along with the bound (2.2) concludes the argument. $\square$

Finally, we make precise the estimate (1.9) referred in the introduction.

Lemma 2.2. *For some $\delta > 0$ the following holds. Let ψ be smooth on $\partial\mathbb{D}$ with $1 \notin \mathrm{supp}(\psi)$. Then*

$$\sum_{x \in \Omega_Q} \psi(x) = \frac{Q^2}{\pi \operatorname{vol}(\Gamma \backslash \mathbb{H})} \int_{\partial\mathbb{D}} \psi(z) \frac{|\mathrm{d}z|}{|1 - z|^2} + O_\psi(Q^{2-\delta}) \qquad (Q > 0).$$

Proof. We split the sum over Ω_Q into cosets modulo the action of $\Gamma_\infty = \left(\begin{smallmatrix} 1 & \omega\mathbb{Z} \\ 0 & 1 \end{smallmatrix}\right)$, getting

$$\sum_{x\in\Omega_Q}\psi(x) = \sum_{x\in G_1\backslash\Omega_Q}\sum_{\tau\in G_1}\psi(\tau x) = \sum_{r\in T_{\infty\mathfrak{b}}(Q\sqrt{\omega})}\sum_{\tau\in\Gamma_\infty}\psi(p_{\mathbb{D}}\tau\omega r) = \sum_{r\in T_{\infty\mathfrak{b}}(Q\sqrt{\omega})}\sum_{n\in\mathbb{Z}}\psi_*(r+n),$$

where $\psi_*(t) := \psi(p_{\mathbb{D}}\omega t)$.

The map $h : \mathbb{R}/\mathbb{Z} \to [0,1]$ defined by $h(r) := \sum_{n\in\mathbb{Z}}\psi_*(r+n)$ is smooth. With the notation [PR18, eq. (3.7)], it satisfies

$$\|h\|_{H^{1/2}} = \sum_{n\in\mathbb{Z}}\left|\widehat{\psi_*}(n)\right|(1+|n|)^{1/2} < \infty$$

where $\widehat{\psi_*}$ is the Fourier transform of ψ_*. We therefore have, by [PR18, Theorem 3.8],

$$\sum_{x\in\Omega_Q}\psi(x) = \sum_{r\in T_{\infty\mathfrak{b}}(Q\sqrt{\omega})} h(r) = \frac{\omega Q^2}{\pi\,\mathrm{vol}(\Gamma\backslash\mathbb{H})}\int_{\mathbb{R}/\mathbb{Z}} h(x)\,\mathrm{d}x + O_\psi(Q^{2-\delta}).$$

Inserting the definition of h and changing variables in the integral yields the claimed main term. $\square$

2.2. **Useful properties of the fundamental domain.** Let R be a fundamental domain in $\mathbb{D}$ under the action of G, whose sides are non-intersecting geodesics. The fact that such a domain R exists is proved in [Tuk73] under the hypotheses (a)-(d). Since R has finite volume, by Siegel's Theorem [Kat92, Theorem 4.1.1], the closure of R in $\overline{\mathbb{D}} = \mathbb{D}\cup\partial\mathbb{D}$ intersects $\partial\mathbb{D}$ at a finite set of points V, and therefore R is an ideal polygon.

Each side s of R is paired with a unique side of R, say $\widehat{s}$, by an element of $g_s \in G\smallsetminus\{1\}$, in such a way that

$$s = g_s\widehat{s}.$$

Under this pairing, we have $\widehat{s} \neq s$. Indeed, if g_s leaves s invariant then it either swaps or fixes its ends. In the first case, g_s would have a fixed point inside $\mathbb{D}$, which contradicts our hypotheses on G. In the second, g_s would fix both ends of s, which are cusps, thus implying that $g_s = 1$, hence a contradiction. Since clearly $\widehat{\widehat{s}} = s$, the map $s \mapsto \widehat{s}$ is thus an involution with no fixed point. In particular, the number of sides of R is even, say $2n$ with $n \geq 2$.

We label the sides of R by the alphabet $\Sigma \simeq \mathbb{Z}/2n\mathbb{Z}$ proceeding counter-clockwise, so that two consecutive elements $i, i+1 \in \mathbb{Z}/2n\mathbb{Z}$ label consecutive sides s_i, s_{i+1}. For $a\in\Sigma$, let s_a be the side associated to a, and define $[a] \subset \partial\mathbb{D}$ be the closed arc whose ends are the vertices delimiting s_a, and which does not intersect other vertices of R. The arcs $\{[a]\}_{a\in\Sigma}$ then form a partition of $\partial\mathbb{D}$ up to vertices.

Given $a \in \Sigma$, define $\widehat{a}$ to be the label associated to $\widehat{s}_a$, and $g_a := g_{s_a}$. We note that $[a] = \overline{g_a(\partial\mathbb{D}\smallsetminus[\widehat{a}])}$. More generally, given a finite word $w \in \Sigma^m$, we let $|w| := m$, and we say that w is admissible if $a_{i+1} \neq \widehat{a_i}$ for all $i \in \{1,\cdots,m-1\}$. The empty word is denoted by ϵ. We define the set of finite admissible words

$$\Sigma^* = \{w \in \cup_{m\geq 0}\Sigma^m, w \text{ is admissible}\}.$$

Given $w = a_1 \dots a_n \in \Sigma^*$, we let $g_w = g_{a_1}\cdots g_{a_n}$. Since G is a free group and the set $\{g_s\}_s$ consists of a basis and its inverses, every $g\in G$ can be written in a unique way as $g = g_w$ for some $w\in\Sigma^*$. Also, for $w\in\Sigma^*$, we define the fundamental interval associated to w as

$$[w] = \begin{cases} \partial\mathbb{D}, & (|w| = 0), \\ [a], & (|w| = 1, w = a), \\ g_{a_1} g_{a_2} \dots g_{a_{n-1}}[a_n], & (w = a_1 \cdots a_n). \end{cases}$$

Note that if $w = aw'$ with $a \in \Sigma$ and $w' \in \Sigma^* \smallsetminus \{\epsilon\}$ not starting with $\widehat{a}$, then $[w] = g_a[w']$. Moreover, an easy induction shows that the admissibility condition ensures $[w] \subsetneq [a]$.

We define next the set of infinite admissible words

$$\Sigma^\omega = \{a_1 a_2 \cdots \mid a_i \in \Sigma, a_{i+1} \neq \widehat{a_i} \ \forall i\}.$$

At this stage, our setting is identical to that described in Sections 2–4 of [Ser81]. Since in our case, the group is free with respect to the generators $\{g_a\}_{a \in \Sigma}$, the graph of G in the terminology of [Ser81] is a tree, and admissible sequences coincide with words in Σ^*. By [Ser81, Proposition 4.5], for any $w = a_1 a_2 \cdots \in \Sigma^\omega$, the intersection

$$[w] := \bigcap_{n \geq 1} [a_1, \dots, a_n]$$

consists of a single point in $\partial\mathbb{D}$, which we still denote by $[w]$.

Define the map

$$\pi : \Sigma^\omega \to \partial\mathbb{D}, \qquad \pi(w) = [w].$$

Lemma 2.3. *The map π is surjective. For all $x \in \partial\mathbb{D}$, we have $|\pi^{-1}(x)| = 1$ except when x belongs to the set of cusps $C(G)$, in which case $|\pi^{-1}(x)| = 2$.*

Proof. This is essentially [Ser81, Proposition 4.6]; with the terminology used there, we are only left to show that the words $w = a_1 a_2 \cdots \in \Sigma^\omega$ which are extreme left or right paths are precisely those for which $\pi(w) \in C(G)$.

Suppose that $\pi(w) \in C(G)$. Then $\pi(w)$ is equivalent to one of the vertices of R, say x_0. Write $\pi(w) = g x_0$ with $g \in G$, and write $g = g_{a_1} \dots g_{a_n}$. We pick x_0 and g such that n is minimal. This ensures that $x_0 \notin [\widehat{a}_n]$, otherwise $x_0' = g_{a_n} x_0$ is also a vertex of R, and the expression $\pi(w) = g' x_0'$ with $g' = g_{a_1} \dots g_{a_{n-1}}$ contradicts the minimality of g. Since x_0 is a vertex of R, which is an ideal polygon, it can be reached through two extreme paths w_L, w_R, which start with consecutive letters, neither of which is equal to $\widehat{a}_n$. Then the words $a_1 \cdots a_n w_L$ and $a_1 \cdots a_n w_R$ are admissible, distinct, and by construction $\pi(w) = \pi(w_L) = \pi(w_R)$. By [Ser81, Proposition 4.6], we get $w \in \{w_L, w_R\}$.

Suppose conversely that w is an extreme path in Σ^ω. This implies there exist $w_1, w_2 \in \Sigma^*$, $w_2 \neq \epsilon$, such that $w = w_1 w'$ with $w' = w_2 w_2 \cdots$. Clearly $g_{w_2}[w'] = [w']$ and thus $[w'] \in C(G)$ and so also $\pi(w) = g_{w_1}[w'] \in C(G)$. $\square$

We also note a cycle relation, which describes the relations among the $2n$ vertices V of R in terms of side-pairings. Choose a starting vertex v_0 and let s_0 be the side of R which ends in v_0 and comes before v_0 in clockwise order. Then let $v_1 = g_{s_0} v_0 \in V$. We repeat this process until the sequence of vertices eventually repeats

$$v_0 \xrightarrow{g_{s_0}} v_1 = g_{s_0} v_0 \xrightarrow{g_{s_1}} v_2 \xrightarrow{g_{s_2}} \cdots \xrightarrow{g_{s_{r-1}}} v_r = v_0.$$

In particular, $g = g_{s_0} \cdots g_{s_{r-1}}$ fixes v_0 and in fact, if $r \geq 1$ is minimal, it generates G_{v_0}. Clearly the vertices $v_0, \dots, v_{r-1}$ are equivalent and the cycles for each of them is obtained rotating the one for v_0. We also remark that choosing the sides after the vertices would generate the same sequence in the opposite direction with corresponding matrix g^{-1}.

Example 2.4. Consider the principal congruence subgroup

$$\Gamma = \Gamma(3) = \{\gamma \in \mathrm{SL}(2,\mathbb{Z}), \gamma \equiv \mathrm{id} \pmod 3\}.$$

Its conjugate inside $\mathrm{SU}(1,1)$ is

$$G(3) = \left\{ g = \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix}, \alpha, \beta \in \tfrac{1}{2}\mathbb{Z} + \tfrac{3i}{2}\mathbb{Z}, |\alpha|^2 - |\beta|^2 = 1 \right\}.$$

A computation by SageMath [Sag23] provides a fundamental domain for $G(3)$ consisting of the ideal polygon whose vertices are given by

$$p_{\mathbb{D}}(\{0, \tfrac{1}{2}, 1, 2, 3, \infty\}) = \{-1, \tfrac{-3-4i}{5}, -i, \tfrac{3-4i}{5}, \tfrac{4-3i}{5}, 1\}.$$

The set Σ has six letters, $\{a, b, c, \widehat{a}, \widehat{b}, \widehat{c}\}$, and we may use the pairing matrices

$$g_a = \begin{pmatrix} 1+\frac{3}{2}i & -\frac{3}{2}i \\ * & * \end{pmatrix}, \quad g_b = \begin{pmatrix} \frac{7}{2}-3i & \frac{9}{2} \\ * & * \end{pmatrix}, \quad g_c = \begin{pmatrix} 1-3i & 3 \\ * & * \end{pmatrix}.$$

Observe that $p_{\mathbb{D}}(1), p_{\mathbb{D}}(\infty)$ are inequivalent to other vertices, whereas the other vertices are related by the following two cycle relations:

$$p_{\mathbb{D}}(0) \to p_{\mathbb{D}}(3) \to p_{\mathbb{D}}(0) \to \cdots, \qquad p_{\mathbb{D}}(1/2) \to p_{\mathbb{D}}(2) \to p_{\mathbb{D}}(1/2) \to \cdots.$$

In Figure 1, we have the drawing of the fundamental domain, hilighting also the set K described in Section 3.1 below.

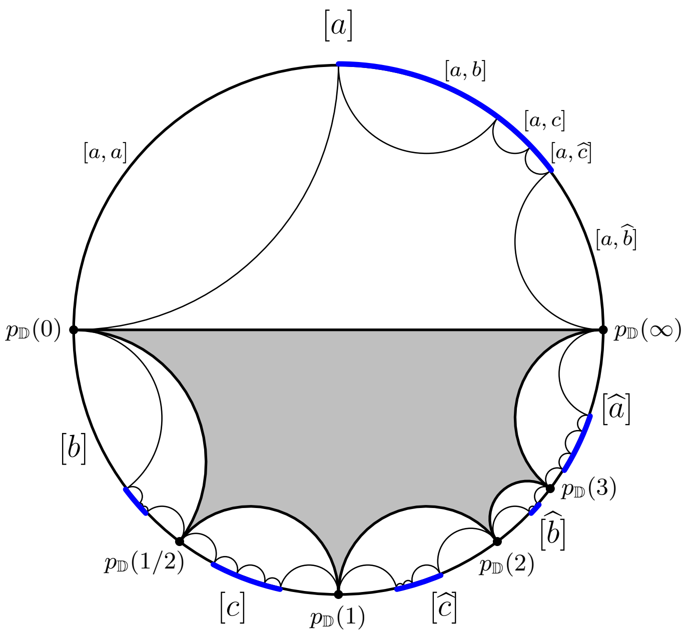


FIGURE 1. Disk model for $\Gamma(3)$ with shaded fundamental domain and highlighting K (in blue). The level 2 subarcs of $[a]$ are also indicated.

2.3. **Bowen-Series map.** For all $x \in \partial\mathbb{D}' = \partial\mathbb{D} \setminus V$, there is a unique $a \in \Sigma$ such that $x \in [a]$. Define the Bowen-Series map

$$\mathcal{B} : \partial\mathbb{D}' \to \partial\mathbb{D}$$

by letting $\mathcal{B}(x) := g_a^{-1}x$, where a is determined by $x \in [a]$.

In the terminology of [Ser81], this map can be seen as the shift map on the subset $\Sigma' \subset \Sigma^\omega$ of words which are not extreme paths starting from 0. Further, the proof of Theorem 4.12 therein, show that $\mathcal{B}$ is a Markov map with respect to the partition $\partial\mathbb{D} = \cup_{a\in\Sigma}[a]$.

3. Induced map and cuspidal accelerations

3.1. **An induced map.** We say that two letters $a, b \in \Sigma$ are separated (or that a is separated from b, or that $|a-b|>1$) if $a-b \notin \{-1,0,1\}$, where we recall that Σ is identified with $\mathbb{Z}/2n\mathbb{Z}$. Following [BS79, p. 164], we define the set K by letting

$$K_a := \bigcup_{\substack{b\in\Sigma \\ |b-\widehat{a}|>1}} [a,b], \qquad K := \bigcup_{a\in\Sigma} K_a.$$

In concrete terms, K can be described as follows: for each $a \in \Sigma$, the map g_a maps $\partial\mathbb{D} \smallsetminus [\widehat{a}]$ into $[a]$, and the image of V under g_a partitions $[a]$ into $2n-1$ intervals. From these intervals, retain the $2n-3$ intervals in the middle, which do not meet V. The union of these middle intervals is K_a.

Given an admissible word $w = a_1 \cdots a_n \in \Sigma^*$ with $n \geq 1$, we call the *K-length* of w the quantity

$$\ell_K(w) := 1 + |\{1 \leq i < n, |a_{i+1} - \widehat{a}_i| > 1\}|. \tag{3.1}$$

and set $\ell_K(\epsilon) := 0$.

Let $\mathcal{D} \subset \partial\mathbb{D}'$ be the set consisting of $C(G) \smallsetminus V$ along with those points $x \notin C(G)$ for which $\mathcal{B}^r(x) \in K$ for some $r \geq 1$. We define the induced map of $\mathcal{B}$ on $K \cup V$ as

$$\mathcal{F}_K : \mathcal{D} \to K \cup V,$$

by

$$\mathcal{F}_K(x) := \mathcal{B}^j(x), \quad j = \min\{t \geq 1, \mathcal{B}^t(x) \in K \cup V\}.$$

By [Sta04, Proposition 4.2], the set $\mathcal{D}$ is of full Lebesgue measure in $\partial\mathbb{D}$.

We will index the inverse branches of $\mathcal{F}_K$ by elements of the following set

$$\Lambda_K(1) := \{w = a_1 \cdots a_n \in \Sigma^*, n \geq 1, \ell_K(a_2 \cdots a_n) \leq 1\}.$$

By definition, this means that given $w \in \Lambda_K(1)$ and $x \in K_b$, with $b \in \Sigma \smallsetminus \{\widehat{a}_n\}$ and $|\widehat{a}_n - b| = 1$ if $n = |w| \geq 2$, then

$$\mathcal{F}_K(g_{a_1\cdots a_n} x) = x.$$

Notice that the condition on b is equivalent to $\ell_K(wb) = 1$, where this condition is meant to include also the requirement that $wb \in \Sigma^*$, i.e. wb is admissible. For all $w \in \Lambda_K(1)$, $w = a_1 \cdots a_n$, we let

$$D_K(w,1) := \bigcup_{\substack{b\in\Sigma, \\ \ell_K(a_2\cdots a_n b)=1}} K_b = \begin{cases} \displaystyle\bigcup_{b\neq\widehat{a}_n} K_b, & (n=1), \\ \displaystyle\bigcup_{|b-\widehat{a_n}|=1} K_b, & (n \geq 2). \end{cases}$$

Note that for $w \neq \epsilon$, we have $D_K(w,1) \neq \varnothing$ if and only if $w \in \Lambda_K(1)$. For all $w \in \Lambda_K(1)$, we define

$$h_{w,1} : D_K(w,1) \to \partial\mathbb{D}, \qquad h_{w,1}(x) = g_{a_1} \ldots g_{a_n} x, \tag{3.2}$$

and

$$\mathcal{H} := \{h_{w,1}, w \in \Lambda_K(1)\}. \tag{3.3}$$

The set $\mathcal{H}$ is the set of inverse branches of $\mathcal{F}_K$.

More generally, for any $r \geq 1$, the set of inverse branches of order $r \geq 1$ of $\mathcal{F}_K$ is parametrized by

$$\Lambda_K(r) := \{w = a_1 \cdots a_n \in \Sigma^*, n \geq r : \ell_K(a_2 \cdots a_n) \in \{r-1, r\}\}.$$

Let

$$D_K(w,r) := \bigcup_{\substack{b\in\Sigma,\\ \ell_K(a_2\cdots a_n b)=r}} K_b,$$

so that $D_K(w,r) = \varnothing$ if and only if $w \in \Lambda_K(r)$, and

$$h_{w,r} : D_K(w,r) \to \partial\mathbb{D}, \qquad h_{w,r}(x) = g_{a_1}\dots g_{a_n}x,$$

where $w = a_1\cdots a_n$. See Figure 2c for an example of $D_K(w,r)$. Finally let

$$\mathcal{H}^r := \{h_{w,r} : w \in \Lambda_K(r)\}$$

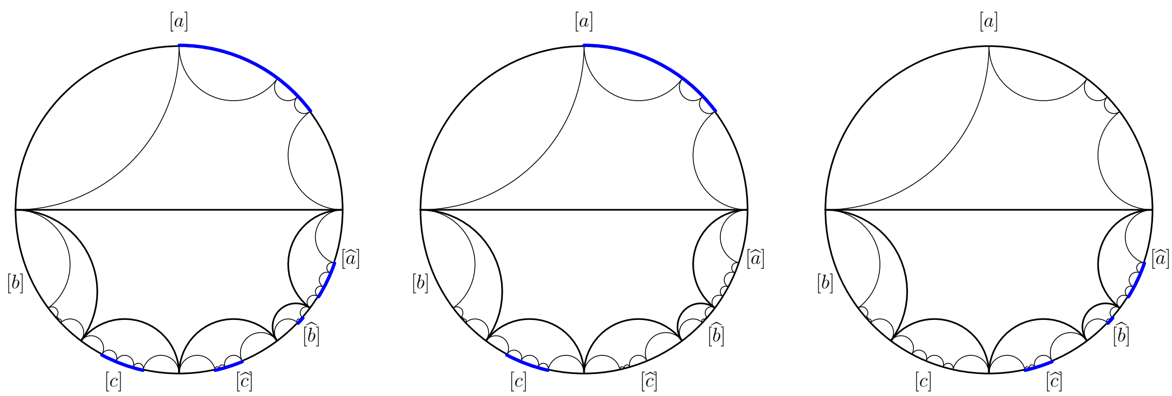
with the notation (3.2); in particular, $\mathcal{H} = \mathcal{H}^1$. We remark that if $\ell_K(a_2\cdots a_n) = r-1$, then the word $w = a_1\cdots a_n$ belongs both to $\Lambda_K(r)$ and $\Lambda_K(r-1)$. The domains of the corresponding inverse branches $h_{w,n}$ and $h_{w,n-1}$ however are different and in fact disjoint.

In any case, given $w \in \Lambda_K(r)$, we will denote from now on

$$\begin{aligned} \mathrm{Dom}(h) &= D_K(w,r) \qquad (h = h_{w,r}),\\ \mathrm{Im}(h) &= h(\mathrm{Dom}(h)). \end{aligned}$$

Finally, we denote $\mathcal{H}^0 = \{\mathrm{id}\}$ and

$$\mathcal{H}^* := \bigcup_{r\geq 0}\mathcal{H}^r.$$



(A) $n = 1$ (whence $r = 1$). (B) $n > 1$ and $\ell_K(w') = r-1$. (C) $n > 1$ and $\ell_K(w') = r$.

FIGURE 2. Domain $D_K(w,r)$ of $h_w \in \mathcal{H}^r$ (underlined in blue), when the last letter of $w = a_1\cdots a_n$ is $a_n = \widehat{b}$, depending on the value of n and $\ell_K(w')$, where $w' = a_2\cdots a_n$. If $r = 1$ only cases (A) and (C) appear, since $\ell_K(w') = 0$ implies $n = 1$.

Lemma 3.1.

(1) For all $x \in K$ and $r \in \mathbb{Z}_{\geq 0}$, we have

$$\mathcal{F}_K^{-r}(x) = \{h(x) : h \in \mathcal{H}^r, x \in \mathrm{Dom}(h)\},$$

and the values $\{h(x)\}$ in the right-hand side are distinct.

(2) For all $h \in \mathcal{H}^r$, we have a unique decomposition in the form $h = h_1 \circ \cdots \circ h_r$, where $h_j \in \mathcal{H}$ and $\mathrm{Im}(h_{i+1}) \subset \mathrm{Dom}(h_i)$.

Proof. Let $x \in K_b$ for some $b \in \Sigma$, and $r \geq 1$. Then by definition, we have $\mathcal{F}_K^r(y) = x$ if and only if y can be expressed as

$$y = g_{w_1} g_{w_2} \cdots g_{w_r} x,$$

where $w_i = a_{i,1} a_{i,2} \cdots a_{i,n_i}$ with $n_i \geq 1$ and :

- $|a_{i,j} - a_{i,j+1}| = 1$ ($1 \leq i \leq r$, $2 \leq j \leq n_i - 1$),
- $|a_{i,n_i} - a_{i+1,1}| = 1$ ($1 \leq i \leq r-1$, $n_i \geq 2$),
- $|a_{r,n_r} - b| = 1$ ($n_r \geq 2$),
- $|a_{i,1} - a_{i,2}| > 1$ ($2 \leq i \leq r$, $n_i \geq 2$),
- $|a_{i,1} - a_{i+1,1}| > 1$ ($2 \leq i \leq r$, $n_i = 1$).

Let $w = w_1 \cdots w_r$, and let $w' = a_{1,2} \cdots a_{1,n_1} w_2 \cdots w_r$ be obtained from w by removing the first letter. Moreover, in the word $w'b$, there are exactly $r-1$ pairs of separated consecutive letters, which are the following pairs:

- $(a_{i,1}, a_{i,2})$ for $2 \leq i \leq r$ where $n_i \geq 2$,
- $(a_{i,1}, a_{i+1,1})$ for $2 \leq i < r$ where $n_i = 1$,
- $(a_{r,1}, b)$ if $n_r = 1$.

We deduce $\ell_K(w'b) = r$, and therefore also $\ell_K(w') \in \{r-1, r\}$ (depending on whether $n_r = 1$ or $n_r \geq 2$). Since clearly we also have $|w| \geq r$, we conclude that $x \in D_K(w,r)$ and $w \in \Lambda_K(r)$. This shows that $y = h(x)$ for $h = h_{w,r} \in \mathcal{H}^r$.

Conversely, let $h \in \mathcal{H}^r$, and write $h = h_{w,r}$ for some $w \in \Lambda_K(r)$. Assume $x \in D_K(w,r)$. This means $\ell_K(w'b) = r$, where w' is obtained from w by removing the first letter. We break

$$w = w_1 w_2 \cdots w_r, \tag{3.4}$$

where $w_2, \ldots, w_r$ start at the first letter of a separated pair in $w'b$. Then we easily see by induction that for all $1 \leq i \leq r$,

$$\mathcal{F}_K^i(g_{w_1} \cdots g_{w_r} x) = g_{w_{1+i}} \ldots g_{w_r} x$$

where the product on the right-hand side is empty if $i = r$. For $i = r$ this shows that $h(x) \in \mathcal{F}_K^{-r}(x)$.

Finally, suppose that $w_1, w_2 \in \Lambda_K(r)$ are such that $x \in D_K(w_1, r) \cap D_K(w_2, r)$ and $y := h_{w_1,r}(x) = h_{w_2,r}(x)$. Letting $b \in \Sigma$ be such that $x \in K_b$, we have that y belong to the arcs $[w_1 b]$ and $[w_2 b]$, which implies (up to swapping w_1 and w_2) that $w_1 b$ is a prefix of $w_2 b$. Let w_i' be obtained from w_i by removing the first letter. Since $x \in K_b$, we have $x \in [b,c]$ with (b,c) separated, and this implies that $\ell_K(w_1' bc) = r+1$. If $w_1 \neq w_2$, then $w_1' bc$ is a prefix of $w_2' b$, and we deduce therefore $\ell_K(w_2' b) \geq r+1$, a contradiction. We conclude that $w_1 = w_2$. This shows that the values $\{h(x), h \in \mathcal{H}^r, x \in \mathrm{Dom}(h)\}$ are distinct.

We now move to proving (2). Let $h \in \mathcal{H}^r$ and $x \in \mathrm{Dom}(h)$, say $x \in K_b \in \Sigma$. Write $h = h_{w,r}$ where $w \in \Lambda_K(r)$ and $x \in D_K(w,r)$, let $w_1, \ldots, w_r$ be constructed as in (3.4), and let b_i be the first letter of w_i. Finally let $b_{i+1} := b$. By construction, we have that within each word $w_i b_{i+1}$, the only possible consecutive pair of separated letters are the first two, and therefore $w_i \in \Lambda_K(1)$ for all i, and also $[b_{i+1}] \subset D_K(w,1)$. By a descending induction, we deduce $g_{w_i} \cdots g_{w_r} K_b \subset [b_i]$ for $1 \leq i \leq r$. Let $h_i := h_{w_i,1} \in \mathcal{H}$. We deduce that $K_b \subset \mathrm{Dom}(h_r)$, and for all $1 \leq i \leq r-1$, that $\mathrm{Im}(h_{i+1}) \subset \mathrm{Dom}(h_i)$. This shows the existence of the decomposition $h = h_1 \circ \cdots \circ h_r$. The uniqueness is easily established. $\square$

3.2. **Cuspidal acceleration.** Most of our analysis will be carried out using the map $\mathcal{F}_K$ in the previous section. For our later arguments and for some applications, however, it will be crucial for us to consider another acceleration of $\mathcal{B}$, which is slower, but whose inverse branches are easier to describe. What follows is inspired by the notion of cuspidal decomposition introduced in the work [AMU16] (see also [AMU20, Mar18]).

Given a word $w = a_1 \cdots a_r \in \Sigma^* \setminus \{\epsilon\}$, we say that w is *cuspidal* if there is $\delta \in \{\pm 1\}$ such that $\widehat{a}_i = a_{i+1} + \delta$ for all $i = 1, \ldots, r-1$. If $\delta = 1$, we say that w is *right-cuspidal*, and if $\delta = -1$ we say that w is *left-cuspidal*. Note that if $|w| = 1$ then w is both left and right-cuspidal (and conversely). Note also that w is cuspidal if and only if the arcs $[a_1], [a_1, a_2], \ldots, [a_1, \ldots, a_r]$ share an endpoint (the right one if w is right-cuspidal and the left one otherwise). These notions coincide with the corresponding notions in [AMU16]. We say that w is a *cuspidal cycle* if wa_1 is cuspidal and that a cuspidal cycle is *minimal* if it has no prefix which is a cuspidal cycle. Notice there are exactly $2|\Sigma|$ minimal cuspidal cycles.

Given $w \in \Sigma^*$ we define the number of *cuspidal flips* $\ell_c(w)$ of w as follows. If $w = \epsilon$ let $\ell_c(w) = 0$, whereas for $w = a_1 \cdots a_n \in \Sigma^*$ with $n \geq 1$, we let

$$\ell_c(w) := 1 + |\{1 \leq i < n-1, \delta_i \delta_{i+1} \neq 1\}|. \tag{3.5}$$

with $\delta_i = 0$ if $|a_{i+1} - \widehat{a}_i| \neq 1$ and otherwise $\delta_i := a_{i+1} - \widehat{a}_i \in \{\pm 1\}$. Notice that, if we denote $\widehat{w} := \widehat{a}_n \ldots \widehat{a}_1$, we have

$$\ell_c(w) = \ell_c(\widehat{w}). \tag{3.6}$$

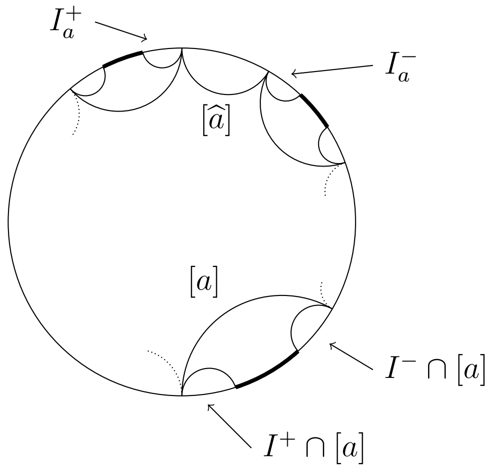
In [AMU16, Definition 4.4], the authors introduce the notion of decomposition into maximal cuspidal words. Note that the quantity $\ell_c(w)$ defined here *does not* correspond, in general, to the number of cuspidal blocks in such a decomposition of w. For instance if $w = a_1 a_2 b_1 b_2$ with $a_1 a_2$ and $b_1 b_2$ right-cuspidal, and $a_2 b_1$ left-cuspidal, then $(\delta_i)_{1 \leq i \leq 3} = (1, -1, 1)$, and therefore $\ell_c(w) = 3$; however the decomposition into cuspidal blocks as defined in [AMU16] has two blocks, namely $w = (a_1 a_2)(b_1 b_2)$.

Before we actually define the accelerated map that will be of interest for us, we first discuss the cuspidal acceleration defined in [AMU16]. This is defined as the map $\mathcal{F}_{\mathrm{AMU}}$ sending $x = [a_1, a_2, \cdots] \in \mathcal{D}$ to $\mathcal{F}_{\mathrm{AMU}}(x) = [a_{r+1}, a_{r+2}, \cdots]$ where $r \geq 1$ is the maximal index such that $a_1 \cdots a_r$ is cuspidal. We let

$$I^{\pm} := \bigcup_{a \in \Sigma} [a, \widehat{a} \pm 1]^*;$$

$$I_a^{\pm} := [\widehat{a} \pm 1, \widehat{\widehat{a} \pm 1} \pm 1],$$

$$I_x^{\pm} := I_a^{\pm}, \qquad (x \in [a] \setminus V, a \in \Sigma),$$



where * indicates that the border of the arc $[a, \widehat{a} \pm 1]$ belonging also to K_a is removed. Then $\mathcal{F}_{\mathrm{AMU}} : \mathcal{D} \to \partial \mathbb{D}$ can be defined as

$$\mathcal{F}_{\mathrm{AMU}}(x) := \begin{cases} \mathcal{B}(x), & \text{if } x \in K \\ \mathcal{B}^{r+1}(x), & \text{if } x \in I^{\pm}, \text{ where } r = \min\{t \geq 1, \mathcal{B}^t(x) \notin I^{\pm}_{\mathcal{B}^{t-1}(x)}\}. \end{cases}$$

We can also alternatively express $\mathcal{F}_{\mathrm{AMU}}$ in terms of ℓ_c as

$$\mathcal{F}_{\mathrm{AMU}}(x) := \begin{cases} \mathcal{B}(x), & \text{if } x \in K, \\ [a_{j+1}, a_{j+2}, \dots], & \text{if } x \notin K, \text{where } j \geq 1 \text{ is maximal such that } \ell_c([a_1 \cdots a_j]) = 1. \end{cases}$$

For our purposes, it is however more natural to consider a variation of $\mathcal{F}_{\mathrm{AMU}}$ which we denote by $\mathcal{F}_c$, defined for $x \in \partial\mathbb{D}'$ in the following way :

$$\mathcal{F}_c(x) := \begin{cases} \mathcal{B}(x), & \text{if } x \in K \\ \mathcal{B}^r(x), & \text{if } x \in I^{\pm}, \text{ where } r = \min\{t \geq 1, \mathcal{B}^t(x) \notin I^{\pm}_{\mathcal{B}^{t-1}(x)}\}. \end{cases} \tag{3.7}$$

The difference, with respect to $\mathcal{F}_{\mathrm{AMU}}$, is that when $x \notin K$ the exponent in the iteration is one less. We remark that the function $\mathcal{F}_K$ defined in the previous section is also the induced map of $\mathcal{F}_c$ on $K \cup V$; this is not true for the map $\mathcal{F}_{\mathrm{AMU}}$.

Similarly as $\mathcal{F}_{\mathrm{AMU}}$, the map $\mathcal{F}_c$ can also be expressed by means of ℓ_c, in the following way : if $x = [a_1, a_2, \dots]$, then

$$\mathcal{F}_c(x) = [a_{j+1}a_{j+2}\cdots], \quad \text{with } j \geq 1 \text{ maximal such that } \ell_c([a_1 \cdots a_{j+1}]) = 1. \tag{3.8}$$

We now classify the inverse branches of $\mathcal{F}_c$. Similarly as for $\mathcal{F}_K$ we parametrize the inverse branches of order $r \geq 1$ of $\mathcal{F}_c$ by the following set of words,

$$\Lambda_c(r) := \{w \in \Sigma^* : \ell_c(w) \in \{r-1, r\},\ |w| \geq r, [\widehat{w}] \not\subset K \text{ if } \ell_c(w) = r\}.$$

Notice that $[\widehat{w}] \not\subset K$ is always satisfied if $|w| = 1$ and if $|w| \geq 2$ it is equivalent to the last two letters of w not being separated. If $r = 1$ we write

$$\Lambda_c := \Lambda_c(1) = \{w \in \Sigma^* : \ell_c(w) = 1, [\widehat{w}] \not\subset K\}.$$

We also let

$$D_c(w, r) := \bigcup_{\substack{b_1, b_2 \in \Sigma \\ \ell_c(wb_1) = r \\ \ell_c(wb_1b_2) = r+1}} [b_1, b_2].$$

Note that $w \in \Lambda_c(r)$ if and only if $D_c(w, r) \neq \varnothing$. We define the corresponding inverse branches through

$$h^c_{w,r} : D_c(w, r) \to \partial\mathbb{D}, \qquad h^c_{w,r}(x) := g_{a_1} \cdots g_{a_n} x \tag{3.9}$$

if $w = a_1 \cdots a_n$. Finally, let $\mathcal{H}^0_c = \{\mathrm{id}\}$,

$$\mathcal{H}^r_c := \{h^c_{w,r}, w \in \Lambda_c(r)\}, \qquad \mathcal{H}^*_c := \bigcup_{r \geq 0} \mathcal{H}^r_c. \tag{3.10}$$

Given $h \in \mathcal{H}^r_c$, we let $\mathrm{Dom}(h)$ and $\mathrm{Im}(h)$ be the domain and range of h, so that $\mathrm{Dom}(h) = D_c(w, r)$ if $h = h^c_{w,r}$. Finally, we drop the dependency on r from the notation when $r = 1$, as was the case for $\mathcal{H}$.

We record the properties of the above discussion in the following lemma. The details of the proof being similar to Lemma 3.1, we leave the details to the reader.

Lemma 3.2. *For all $x \in \partial\mathbb{D}$, we have*

$$\mathcal{F}_c^{-r}(x) = \{h(x) : h \in \mathcal{H}^r_c, x \in \mathrm{Dom}(h)\},$$

and the values $\{h(x)\}$ in the right-hand side are distinct.

We will also need to consider the inverse branches of $\mathcal{H}_c$ with a restricted domain. Given $w \in \Lambda_c$, define

$$D_c^\sharp(w) := \bigcup_{\substack{b_1 \in \Sigma,\, b_2 \in \{\widehat{b_1} \pm 1\}, \\ \ell_c(wb_1)=1 \\ \ell_c(wb_1b_2)=2}} [b_1, b_2] \subset D_c(w),$$

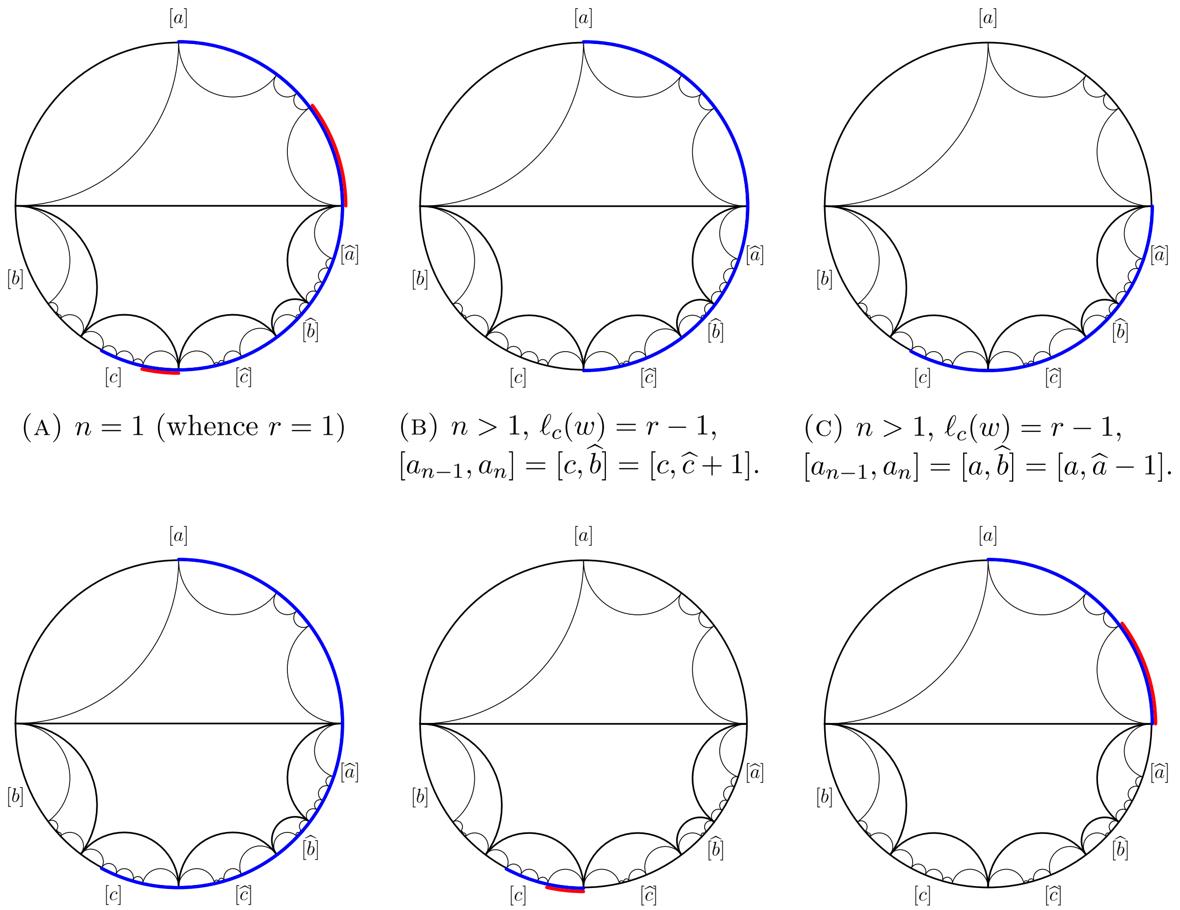
and

$$h_w^{c\sharp} := h_w^c|_{D_c^\sharp(w)}.$$

Finally, let

$$\mathcal{H}_c^\sharp := \{h_w^{c\sharp}, w \in \Lambda_c\}. \tag{3.11}$$

We remark that the same word $w \in \Sigma^*$ can index elements of $\mathcal{H}$, $\mathcal{H}_c$ and $\mathcal{H}_c^\sharp$. While the corresponding inverse branch h_w is defined by the same Möbius transformation in all cases, its domain does depend on whether we consider h_w to be an element of $\mathcal{H}$, $\mathcal{H}_c$ or $\mathcal{H}_c^\sharp$.



(A) $n = 1$ (whence $r = 1$)

(B) $n > 1$, $\ell_c(w) = r - 1$, $[a_{n-1}, a_n] = [c, \widehat{b}] = [c, \widehat{c} + 1]$.

(C) $n > 1$, $\ell_c(w) = r - 1$, $[a_{n-1}, a_n] = [a, \widehat{b}] = [a, \widehat{a} - 1]$.

(D) $n > 1$, $\ell_c(w) = r - 1$, a_{n-1}, a_n separated.

(E) $n > 1$, $\ell_c(w) = r$, $[a_{n-1}, a_n] = [c, \widehat{b}] = [c, \widehat{c} + 1]$.

(F) $n > 1$, $\ell_c(w) = r$, $[a_{n-1}, a_n] = [a, \widehat{b}] = [a, \widehat{a} - 1]$.

FIGURE 3. Domain $D_c(w, r)$ of $h_w \in \mathcal{H}_c^r$ (underlined in blue), where the last letter of $w = a_1 \cdots a_n$ is $a_n = \widehat{b}$, depending on the value of n and $\ell_c(w)$ and a_{n-1}. If $r = 1$ only case (A), (E) and (F) appear. The restricted domain $D_c^\sharp(w)$ (for $r = 1$) is highlighted in red.

The relevance of $\mathcal{H}_c^\sharp$ can be seen from the following simple lemma, which characterizes inverse branches $\mathcal{H}$ for $\mathcal{F}_K$ in terms of inverse branches $\mathcal{H}_c, \mathcal{H}_c^\sharp$ for $\mathcal{F}_c$.

Lemma 3.3. *Let $x \in K$. Then the set*

$$\{h \in \mathcal{H}, x \in \mathrm{Dom}(h)\}$$

is in bijection with the set

$$\bigcup_{r\geq 1}\left\{(h_1,\dots,h_r) : \begin{cases} h_1,\dots,h_{r-1} \in \mathcal{H}_c^\sharp \text{ and } h_r \in \mathcal{H}_c, \\ \mathrm{Im}(h_{i+1}) \subset \mathrm{Dom}(h_i), (1 \leq i \leq r-1), \\ x \in \mathrm{Dom}(h_r). \end{cases}\right\},$$

in such a way that corresponding elements satisfy

$$h(x) = h_1 \circ \cdots \circ h_r(x). \tag{3.12}$$

Proof. Write $x \in [c_1, c_2] \subset K_{c_1}$ for some $c_1, c_2 \in \Sigma$. We rephrase the desired correspondence in terms of the associated words writing $h = h_{w,1}$ and $h_i = h_{w_i}^{c\sharp}$ $(i \leq r-1)$ and $h_r = h_{w_r}^c$. We need to prove that sets

$$M_1 = \{w \in \Lambda_K(1), [c_1, c_2] \subset D_K(w, 1)\}$$

and

$$M_2 = \bigcup_{r\geq 1}\left\{(w_1,\dots,w_r), \begin{cases} w_i \in \Lambda_c, \\ [w_{i+1}\cdots w_r c_1 c_2] \subset D_c^\sharp(w_i) \quad (1 \leq i \leq r-1), \\ [c_1, c_2] \subset D_c(w_r). \end{cases}\right\}$$

are in one-to-one correspondance through $w = w_1 \cdots w_r$.

Suppose $(w_1,\dots,w_r) \in M_2$ and let $w = w_1 \cdots w_r$. Additionnally, let w' be obtained from w by removing the first letter. Let (b_1, b_2) be two consecutive letters in w'. If (b_1, b_2) both belong to a single subword w_i, then the condition $w_i \in \Lambda_c$ ensures that $\ell_c(w_i) = 1$ and thus (b_1, b_2) are not separated. If $b_1 \in w_i$ and $b_2 \in w_{i+1}$ for some $i \leq r-1$, the condition $[w_{i+1}\cdots] \subset D_c^\sharp(w_i)$ ensures that $\ell_c(w_i b_2) = 1$, and thus if $|w_i| \geq 2$ then (b_1, b_2) are not separated either. Finally, if $|w_i| = 1$, then since $b_1 = w_i$ is in w' we must have $i \geq 2$ and thus $[w_i \cdots] \subset D_c^\sharp(w_{i-1})$ which implies that $b_1 b_2$ are not separated also in this case. Since this is true for any consecutive pair of letters in w', we conclude that that $\ell_K(w') = 1$, and therefore $w \in \Lambda_K(1)$. Finally, since $[c_1, c_2] \subset D_c(w_r)$, we have in particular $\ell_c(w_r c_1) = 1$, and therefore also $\ell_K(w' c_1) = 1$. This proves that $x \in \mathrm{Dom}(w)$, and thus $w \in M_1$.

Conversely, let $w \in M_1$, which we write $w = a_1 \cdots a_m \in \Sigma^*$ with $a_i \in \Sigma$ and $m \geq 1$. Let also $a_{m+1} := c_1, a_{m+2} := c_2$, and let w' be obtained from w by removing the first letter. By hypothesis, we have $\ell_K(a_2 \cdots a_m c_1) = 1$, which means that all consecutive pairs of letters in $w' c_1$ are non-separated. Let $r := \ell_c(w' c_1) \geq 1$. For $1 \leq i \leq m$, we define

$$\delta_i := \widehat{a_i} - a_{i+1} \in \{\pm 1\}.$$

By definition of r, there are exactly $r-1$ values of $j \in \{2,\dots,m\}$ for which $\delta_{j-1}\delta_j \neq 1$. We let $2 \leq m_2 < \cdots < m_r \leq m$ denote these values, and $m_1 := 1$, $m_{r+1} := m+1$. Finally, for all $i \in \{1,\dots,r\}$ we define the words

$$w_i := a_{m_i} a_{m_i+1} \dots a_{m_{i+1}-1}.$$

We clearly have by construction $|w_i| \geq 1$, and $w = w_1 \cdots w_r$. Moreover, for all $1 \leq i \leq r$, we have $\delta_j \delta_{j+1} = 1$ whenever $m_i \leq j < m_{i+1} - 2$, therefore $\ell_c(w) = 1$ and thus $w_i \in \Lambda_c$.

Next we consider $i \in \{1, \dots, r\}$ and we compute $\ell_c(w_i a_{m_{i+1}})$. If $|w_i| = 1$, then automatically $\ell_c(w_i a_{m_{i+1}}) = 1$. Otherwise, we have $m_i < m_{i+1} - 1$, and therefore by definition $\delta_{m_{i+1}-2}\delta_{m_{i+1}-1} = 1$. This, along with $\ell_c(w_i) = 1$, ensures that $\ell_c(w_i a_{m_{i+1}}) = 1$. In both cases, we have shown that $\ell_c(w_i a_{m_{i+1}}) = 1$. On the other hand, by definition of m_{i+1}, we have $\delta_{m_{i+1}-1}\delta_{m_{i+1}} \neq 1$, and therefore $\ell_c(w_i a_{m_{i+1}} a_{m_{i+1}+1}) = 2$. We have thus shown that $[w_{i+1} \cdots w_r c_1 c_2] \subset D_c(w_i)$. Finally, if $i \leq r - 1$, the condition $\ell_K(w' c_1) = 1$ ensures that $[a_{m_{i+1}}, a_{m_{i+1}+1}] \not\subset K$, and therefore $[w_{i+1} \cdots w_r c_1] \subset D_c^\sharp(w_i)$. This concludes the proof that $(w_1, \dots, w_r) \in M_2$. ☐

Given $y \in \mathcal{D}$, we let

$$r_y := \min\{r \geq 1 \mid \mathcal{F}_c^r(y) \in K \cup V\} \tag{3.13}$$

be the return time of $\mathcal{F}_c$ to K. Clearly, for $h \in \mathcal{H}$ and $x \in \mathrm{Dom}(h)$ we have that

$$r_{h(x)} = r \tag{3.14}$$

with r as in the decomposition (3.12) of h.

It will also be useful to go in the opposite direction, relating the inverse branches $\mathcal{H}$ with those in $\mathcal{H}$ and $\mathcal{H}_c^\sharp$.

Lemma 3.4. *Let $x \in \partial\mathbb{D}$. Then the set*

$$\{h \in \mathcal{H}_c^*, x \in \mathrm{Dom}(h)\}$$

is in bijection with the set

$$\bigcup_{\substack{r,s\geq 0,\\ r+s\geq 1}} \left\{ (h_1, \dots, h_{r+s}) : \begin{cases} h_1, \dots, h_r \in \mathcal{H} \text{ and } h_{r+1}, \dots, h_{r+s} \in \mathcal{H}_c^\sharp, \\ \mathrm{Im}(h_{i+1}) \subset \mathrm{Dom}(h_i), (1 \leq i \leq r+s-1), \\ x \in \mathrm{Dom}(h_{r+s}). \end{cases} \right\},$$

in such a way that corresponding elements satisfy

$$h(x) = h_1 \circ \cdots \circ h_{r+s}(x). \tag{3.15}$$

Remark 3.5. Notice that if $x \in K$, then necessary $s = 0$. Moreover, if $s \geq 1$, then $\mathrm{Im}(h_{r+1}) \subseteq K$.

Proof. Let $h \in \mathcal{H}_c^*$, with $x \in \mathrm{Dom}(h)$. Write $h = h_{w,m}$ with $m \geq 1$ and $w = a_1 \cdots a_\ell \in \Lambda_c(m)$. Let $a_{\ell+1}, a_{\ell+2} \in \Sigma$ be such that $x \in [a_{\ell+1}, a_{\ell+2}] \subset D_c(w, r)$. Let q be the maximum index in $\{1, \dots, \ell\}$ such that $a_1 \cdots a_q \in \Lambda_K(r)$ for some $r \geq 0$, (with $\Lambda_K(0) = \{\epsilon\}$) and $[a_{q+1}, a_{q+2}] \subset D_K(a_1 \cdots a_q, r)$. Notice that by construction

$$[a_i, a_{i+1}] \not\subset K \quad \forall i = q+2, \dots, \ell+1. \tag{3.16}$$

We can then write $w' := a_{q+1} \cdots a_\ell \in \Lambda_c(s)$ for some $s \geq 0$ (with $\Lambda_c(0) = \{\epsilon\}$). Clearly then $h_{w',s} \in \mathcal{H}_c^s$ and the condition (3.16) ensures that $h_{w',s}$ splits into s compatible elements of $\mathcal{H}_c^\sharp$.

The process above gives a map between the two sets in the statements. By construction (3.15) holds, and it is easily verified that the map is a bijection. ☐

3.3. **Comparing the three maps.** We conclude the section with an example to help visualize the difference between the three maps. Consider the example $x = [w]$ with

$$w = a\, a^+\, a^{++}\, b\, b^+\, b^{+-}\, b^{+-+}\, b^{+-++}\, c\, c^-\, d \cdots \in \Sigma^\omega$$

where for all $m \in \Sigma$, we wrote $m^{\pm} := \widehat{m} \pm 1$ and where $|\widehat{a^{++}} - b|, |\widehat{b^{+-++}} - c|, |\widehat{c^{-}} - d| > 1$. Then the first few elements in the orbit of w under the three maps introduced above are as follows:

r	1	2	3	4	5	6	7
$\mathcal{F}_K^r(w)$	$a^{++}b\cdots$	$b^{+-++}c\cdots$	$c^{-}d\cdots$				
$\mathcal{F}_c^r(w)$	$a^{++}b\cdots$	$bb^{+}\cdots$	$b^{+}b^{+-}\cdots$	$b^{+-}b^{+-+}\cdots$	$b^{+-++}c\cdots$	$cc^{-}\cdots$	$c^{-}d\cdots$
$\mathcal{F}_{\mathrm{AMU}}^r(w)$	$bb^{+}\cdots$	$b^{+-}b^{+-+}\cdots$	$cc^{-}\cdots$	$d\cdots$			

where for each point of the orbit we indicated the first letters of the corresponding word in Σ^{ω}.

3.4. **Natural extensions and invariant measures.**

3.4.1. *The invariant measure for $\mathcal{B}$ and $\mathcal{F}_K$.* Let

$$\mathrm{d}s(x) = \frac{\mathrm{d}x}{ix} = \mathrm{d}\theta \qquad (x = \mathrm{e}^{i\theta} \in \partial\mathbb{D})$$

be the arc length measure on the circle. It is a routine computation to check that the measure ζ on $\partial\mathbb{D}^2$ given by

$$\zeta(\mathrm{d}x, \mathrm{d}y) = \frac{2\,\mathrm{d}x\,\mathrm{d}y}{(x-y)^2} = \frac{-2xy\,\mathrm{d}s(x)\,\mathrm{d}s(y)}{(x-y)^2} = \frac{\mathrm{d}s(x)\,\mathrm{d}s(y)}{1 - \mathfrak{Re}(y\overline{x})}$$

is invariant under the transformation $(x, y) \mapsto (gx, gy)$ if $g \in \mathrm{SU}(1,1)$; see e.g. [AS19, p. 528] for the computation on the half-plane model.

Following the argument of [Ser85, Section 3.1] (see also [AF91] and [AS19, Section 6]), we can compute invariant measures for $\mathcal{F}_K, \mathcal{F}_c$ and $\mathcal{F}_{\mathrm{AMU}}$ by integrating $\zeta(\mathrm{d}x, \mathrm{d}y)$ along a well-chosen set.

Proposition 3.6. *For all $x \in \partial\mathbb{D} \smallsetminus V$, define*

$$\xi(x) := \int_{y \in \partial\mathbb{D} \smallsetminus [a]} \frac{\zeta(\mathrm{d}x, \mathrm{d}y)}{\mathrm{d}s(x)} = \int_{y \in \partial\mathbb{D} \smallsetminus [a]} \frac{\mathrm{d}s(y)}{1 - \mathfrak{Re}(y\overline{x})}, \qquad (x \in [a]). \tag{3.17}$$

- *The measure $\xi(x)\,\mathrm{d}s(x)$ on $\partial\mathbb{D}$ is $\mathcal{B}$-invariant.*
- *The measure $\xi(x)\,\mathrm{d}s(x)$ restricted to K is $\mathcal{F}_K$-invariant.*

Proof. See [Sta04, Section 4]. The value of the measure is obtained from a stable region for an associated natural extension, which is denoted Y in [Sta04]. □

Note that $\xi(x)$ takes positive values, and is smooth and bounded outside any neighborhood of V. In particular, we have

$$\xi(x) \asymp 1, \qquad (x \in K).$$

We also notice that by the invariance of $\zeta(\mathrm{d}x, \mathrm{d}y)$ and a change of variable we have

$$|g_w'(x)|\xi(g_w(x)) = \int_{y \in [\widehat{w}]} \frac{\zeta(\mathrm{d}x, \mathrm{d}y)}{\mathrm{d}s(x)} \tag{3.18}$$

for all $w = a_1 \cdots a_m \in \Sigma^*$ and $x \in [a]$ with wa admissible.

3.4.2. *The invariant measures for $\mathcal{F}_c$ and $\mathcal{F}_{\mathrm{AMU}}$.* We proceed in a similar fashion for $\mathcal{F}_c$. For all $a \in \Sigma$ and $x \in [a]$, we define

$$B_x := \begin{cases} [a], & (x \in K_a), \\ [a] \cup [a-1], & (x \in [a, \widehat{a}+1] \smallsetminus K_a), \\ [a] \cup [a+1], & (x \in [a, \widehat{a}-1] \smallsetminus K_a). \end{cases}$$

Then, we define

$$\varphi(x) = \int_{\partial\mathbb{D}\smallsetminus B_x} \frac{\zeta(\mathrm{d}x, \mathrm{d}y)}{\mathrm{d}s(x)}. \tag{3.19}$$

By construction, we have $\inf_{x\in\partial\mathbb{D}} \operatorname{dist}(x, \partial\mathbb{D} \smallsetminus B_x) > 0$, and thus

$$\varphi(x) \asymp 1, \qquad (x \in \partial\mathbb{D}).$$

Notice also that φ is positive, smooth outside of the border of ∂K, and that $\varphi(x) = \xi(x)$ for all $x \in K$.

Proposition 3.7. *The measure $\varphi(x)\,\mathrm{d}s(x)$ is invariant under $\mathcal{F}_c$.*

Proof. Although this can be checked directly, we introduce the underlying natural extension, from which the definition of $\varphi(x)$ was derived.

Let $A \subset \partial\mathbb{D}$ consist of those elements $x = [a_1, a_2, \dots]$ not in $C(G)$, for which $[a_j, a_{j+1}] \subset K$ for infinitely many values of j. This is the recurrence set for $\mathcal{F}_K$, and it is a full measure set [Sta04, Theorem 2]. For all $x \in A$, we let $g_x \in G$ be the element for which $\mathcal{F}_c(x) = g_x^{-1}x$, and we define

$$\mathcal{T}_c(x, y) := (g_x^{-1}x, g_x^{-1}y), \qquad ((x, y) \in A \times A).$$

We define a non-trivial subset of $A \times A$ on which $\mathcal{T}_c$ acts and is invertible. In concrete examples, this set can be guessed numerically using the method in [AS19]. In our case, we define

$$N_0 := \bigcup_{\substack{a_1, a_2, b \in \Sigma \\ \ell_c(\widehat{a}_2\widehat{a}_1 b) = 2}} [a_1, a_2] \times [b] = \bigcup_{x\in\partial\mathbb{D}} \{x\} \times (\partial\mathbb{D} \smallsetminus B_x),$$

and

$$N := N_0 \cap (A \times A).$$

Notice that $\varphi(x) = \int_{y\in\partial\mathbb{D}} \mathbf{1}_{(x,y)\in N} \frac{\zeta(\mathrm{d}x,\mathrm{d}y)}{\mathrm{d}s(x)}$. In order to prove the proposition, following the argument in *e.g.* [Ser85, section 3.1], it suffices to show that $\mathcal{T}_c$ is one-to-one from N to itself.

We construct the inverse $\mathcal{S}$ of $\mathcal{T}_c$ as follows. Let $(x, y) \in N$ with $x = [a_1 a_2 \cdots]$ and $y = [b_1 b_2 \cdots]$. Then, let $j \geq 1$ be the maximal integer such that $\ell_c(\widehat{a}_1 w) = 1$ with $w = b_1 \cdots b_j$. Since $y \in A$ such a j exists. Moreover, since $\ell_c(\widehat{a}_2\widehat{a}_1 b_1) = 2$ it follows that also $\ell_c(\widehat{a}_2\widehat{a}_1 w) = 2$. By (3.6) we have that $x \in D_c(\widehat{w})$ and we let

$$\mathcal{S}(x, y) := (h_{\widehat{w}}(x), g_w^{-1}(y)) = \big([\widehat{b}_j \cdots \widehat{b}_1 a_1 \cdots], [b_{j+1} b_{j+2} \cdots]\big).$$

This belongs to N since, by definition of j, we have $\ell_c(b_{j-1} b_j b_{j+1}) = 2$ (or $\ell_c(\widehat{a}_1 b_1 b_2) = 2$ if $j = 1$). By construction $\mathcal{T}_c \circ \mathcal{S} = \mathrm{id}$. We need to show that also $\mathcal{S} \circ \mathcal{T}_c = \mathrm{id}$. Let $(x, y) \in N$ as above and let $j \geq 1$ be such that $1 = \ell_c(w a_{j+1}) < \ell_c(w a_{j+1} a_{j+2}) = 2$ with $w = a_1 \cdots a_j$. By (3.8) we have

$$\mathcal{T}_c(x, y) = (g_w^{-1}x, g_w^{-1}y) = \big([a_{j+1} a_{j+2} \cdots], [\widehat{a}_j \cdots \widehat{a}_1 b_1 b_2 \cdots]\big).$$

This also belongs to N since $\ell_c(\widehat{a_{j+2}}\widehat{a_{j+1}}\widehat{a}_j) = 2$ by (3.6), and it immediately follows that $(\mathcal{S} \circ \mathcal{T}_c)(x, y) = (x, y)$. $\square$

For completeness, and since it could be of independent interest, we also compute an invariant measure for $\mathcal{F}_{\mathrm{AMU}}$, although it will play no role in the paper.

Given $a \in \Sigma$, let $w_*^{\pm} = [a_1, a_2, \cdots]$ with $a_1 = a \pm 1$ and $a_{j+1} = \widehat{a}_j \pm (-1)^j$ for $j \geq 1$. Notice that $w_*^{\pm}$ are periodic points of the Bowen series map but are not cusps, because the period is not a cuspidal cycle [AMU16, Lemma 4.9]. In particular, they are hyperbolic points.

For $a \in \Sigma$, let Y_a be the interval in $\partial\mathbb{D}$ from w_*^- to w_*^+ in the counter-clockwise direction. We note that Y_a containes a neighborhood of $[a]$. We can write the complement to Y_a in $\partial\mathbb{D}$ as

$$\partial\mathbb{D} \smallsetminus Y_a = \bigcup_{\substack{b_1\cdots b_m \in \Sigma^*,\ m \geq 1 \text{ odd} \\ b_1 = a\pm 1 \Rightarrow b_{j+1} = \widehat{b}_j \pm (-1)^j,\ 1 \leq j < m-1 \\ b_m \neq \widehat{b}_{m-1} \pm (-1)^{m-1},}} [b_1, \ldots, b_m]$$

where b_{m-1} in the last line of the subscript should be interpreted as $b_0 := \widehat{a}$ if $m = 1$, so that when $m = 1$ the condition becomes $|b_1 - a| > 1$.

We let

$$\varphi_M(x) := \int_{\partial\mathbb{D}\smallsetminus Y_a} \frac{\zeta(\mathrm{d}x, \mathrm{d}y)}{\mathrm{d}s(y)} \qquad y \in [a].$$

Since $\operatorname{dist}([a], \partial\mathbb{D} \smallsetminus Y_a) > 0$ for all $a \in \Sigma$, we have $\varphi_M(x) \asymp 1$ for all $x \in \partial\mathbb{D}$.

Proposition 3.8. *The measure $\varphi_M(x)\,\mathrm{d}s(x)$ on $\partial\mathbb{D}$ is invariant under $\mathcal{F}_{\mathrm{AMU}}$.*

Proof. Analogously to the earlier cases, we let $\mathcal{T}_M(x, y) := (g_x^{-1}x, g_x^{-1}y)$ with g_x such that $\mathcal{F}_{\mathrm{AMU}}(x) = g_x^{-1}x$, and we define

$$N_0 := \bigcup_{a\in\Sigma} [a] \times (\partial\mathbb{D} \smallsetminus Y_a),$$
$$N := (A \times A) \cap N_0.$$

Again it suffices to show that the map $\mathcal{T}_M$ is one-to-one on N. Let $(x, y) \in N$ with $x = [a_1a_2\cdots]$ and $y = [b_1b_2\cdots]$. We define

$$\mathcal{S}(x, y) = (g_w^{-1}(x), g_w^{-1}(y)) = \big([\widehat{b}_j \cdots \widehat{b}_1 a_1 \cdots], [b_{j+1}b_{j+2}\cdots]\big),$$

for $w = b_1 \ldots b_j$ and $j \geq 1$ defined as follows.

- If $|b_1 - a_1| > 1$ and $|\widehat{b}_1 - b_2| > 1$, then $j := 1$.
- If $|b_1 - a_1| > 1$ and $b_2 = \widehat{b}_1 + \delta$ for some $\delta \in \{\pm 1\}$, then we let $n \geq 1$ be the minimal integer such that $b_{n+2} \neq \widehat{b}_{n+1} + \delta$. Let $r \geq 1$ be the minimal integer such that $b_{n+r+1} \neq \widehat{b}_{n+r} + \delta(-1)^r$. Then we take $j := n$ if r is even and $j := n + 1$ if r is odd.
- If $b_1 = a_1 \pm 1$, then $j := 2$.

By construction it can be checked that $\mathcal{S}(x, y) \in N$ and $\mathcal{T}_M \circ \mathcal{S}(x, y) = (x, y)$. We leave to the interested reader to verify that $\mathcal{T}_M(N) \subset N$ and $\mathcal{S} \circ \mathcal{T}_M = \mathrm{id}$. $\square$

Remark 3.9. In [Mar18], a technical assumption is made on the fundamental domain of the group (Assumption 2.1 *ibid.*). We believe that the metric estimates on inverse branches which we work out for $\mathcal{F}_c$ in Section 4.2 could be useful to relax this assumption.

4. Metric properties of inverse branches

4.1. **Distortion and expansion estimates.** We quote the following uniform bounded distortion property from [Sta04, Lemma 4.3], see also [BS79, p. 164-165].

Lemma 4.1. *There exists $C \geq 0$ such that for all $h \in \mathcal{H}^*$ we have*

$$|h''(x)| \leq C|h'(x)|, \qquad (x \in \mathrm{Dom}(h)). \tag{4.1}$$

Proof. The set $B = K$ satisfies Lemma 4.3 from [Sta04]. Let $C > 0$ be the value given by this lemma. Let $h \in \mathcal{H}^*$, $h = h_{w,r}$ for some $r \geq 0$. Thus, for any $x \in \mathrm{Dom}(h)$, if $n := |w|$, then $\mathcal{B}^n(h(x)) = x \in \mathrm{Dom}(h) \subset K$. We deduce[2]

$$\left|\frac{h''}{h'}(x)\right| = \left|\frac{D^2(\mathcal{B}^n)}{D(\mathcal{B}^n)^2}(h(x))\right| \leq C$$

where D stands for the derivative, and the last bound is [Sta04, Lemma 4.3]. □

Corollary 4.2. *For $h, k \in \mathcal{H}^* \setminus \{\mathrm{id}\}$ and $x \in \mathrm{Dom}(h) \cap \mathrm{Dom}(k)$, we have*

$$\left|\frac{h''}{h'}(x) - \frac{k''}{k'}(x)\right| \asymp \left|h^{-1}(0) - k^{-1}(0)\right|, \tag{4.2}$$

$$|h'(x)| \asymp |h'(0)|, \tag{4.3}$$

$$|h(x) - h(0)| \ll |h'(x)|. \tag{4.4}$$

Proof. Write

$$h(x) = \frac{\alpha x + \beta}{\overline{\beta} x + \overline{\alpha}}$$

and similarly we denote α', β' the components of k. Since $h \neq \mathrm{id}$, we have $\beta \neq 0$, and since G does not contain elliptic elements, we have $\alpha \neq 0$. The same holds for α', β'. By discreteness of G, we have $\beta \gg 1$, and therefore $|\alpha/\beta| = \sqrt{1 + 1/|\beta|^2} \asymp 1$. The bounded distortion property (4.1) implies in our case

$$\left|x + \overline{\alpha}/\overline{\beta}\right| \asymp 1$$

and similarly for α', β'. Indeed the lower-bound is exactly (4.1), while the upper-bound holds since both x and α/β are bounded. Then,

$$\left|\frac{h''}{h'}(x) - \frac{k''}{k'}(x)\right| = \frac{|\beta/\alpha - \beta'/\alpha'|}{\left|(x + \overline{\beta}/\overline{\alpha})(x + \overline{\beta}'/\overline{\alpha}')\right|^{1/2}} \asymp |\beta/\alpha - \beta'/\alpha'|.$$

This proves the first item. The second item follows by the simple computation

$$|h'(x)| = |h'(0)| \cdot \left|\tfrac{\alpha}{\beta}\right|^2 \cdot \left|x + \overline{\alpha}/\overline{\beta}\right|^{-2} \asymp |h'(0)|.$$

Similarly, the third item follows since

$$|h(x) - h(0)| = |h'(x)| \cdot \left|\tfrac{\beta}{\alpha}\right| \cdot \left|x + \overline{\alpha}/\overline{\beta}\right| \asymp |h'(x)|. \qquad □$$

We proceed to show the uniform expansion property, expressed in terms of inverse branches. This is essentially extracted from [Sta04, Proposition 4.4].

[2] In the proof of [Sta04, Lemma 4.3] it is assumed that $\mathrm{Im}(h) \subset K$, however this is not actually used in the proof, nor is present in the statement of the Lemma. See also Proposition 4.8 below.

Lemma 4.3. *For some $\rho \in [0,1)$, all $r \geq 1$ and $h \in \mathcal{H}^r$, we have*

$$|h'(x)| \ll \rho^r \qquad \forall x \in \mathrm{Dom}(h) \tag{4.5}$$

with an absolute implied constant.

Proof. Let $h \in \mathcal{H}^r$, $h = h_{w,r}$ with $w \in \Lambda_K(r)$. Then proceeding as in [Aar97, p. 145] we have

$$\mathrm{meas}(\mathrm{Dom}(h))|h'(x)| = \int_{\mathrm{Dom}(h)} |h'(x)| \,\mathrm{d}s(y) \ll \int_{\mathrm{Dom}(h)} |h'(y)| \,\mathrm{d}s(y) = \mathrm{meas}(\mathrm{Im}(h))$$

where we used (4.3) and meas denotes the measure on $\partial\mathbb{D}$. We have $\mathrm{meas}(\mathrm{Dom}(h)) \gg 1$, and therefore

$$|h'(x)| \ll \mathrm{meas}([w]).$$

This implies at once

$$\|h'\|_\infty \ll 1 \qquad (h \in \mathcal{H}^r). \tag{4.6}$$

We then argue as in [Sta04, Prop. 4.4]: By [Ser81, Prop. 4.5], $\sup_{|w|=n} \mathrm{meas}([w]) = o(1)$ as $n \to \infty$. Therefore, since $|w| \geq r$, we may find $r_0 \geq 1$ such that

$$\rho := \sup_{h \in \mathcal{H}^{r_0}} \|h'\|_\infty < 1. \tag{4.7}$$

Then choosing $r \geq 1$ arbitrary, we write $r = qr_0 + s$ with $q \geq 0$, $0 \leq s < r_0$. We write accordingly

$$h = h_1 \circ \cdots \circ h_q \circ h_{q+1},$$

where $h_j \in \mathcal{H}^{r_0}$ for $j \leq q$ and $h_{q+1} \in \mathcal{H}^s$ (or $h_{q+1} = \mathrm{id}$ if $s = 0$). We deduce

$$\|h'\|_\infty \leq \prod_j \|h_j'\|_\infty \ll \rho^q \ll (\rho^{1/r_0})^r$$

where we have used (4.6) for $j = q+1$ and (4.7) for $j \leq q$. Reinterpreting ρ gives our claim. □

4.2. **Estimates for cuspidal inverse branches.** When studying the convergence of the relevant transfer operator, we shall need an estimate for the derivatives of the elements of $\mathcal{H}_c$. First, given a cuspidal cycle w, we describe $[w^\infty] := [w, w, \dots]$ and its stabilizer.

Lemma 4.4. *Let $w = a_1 \cdots a_r$ be a cuspidal cycle. Then:*

(1) The point $[w^\infty] = [w, w, \dots]$ belongs to V; it is the right-end boundary of $[a_1]$ if w is right-cuspidal, and otherwise it is the left-end boundary of $[a_1]$.
(2) The element $g_w \in G$ is parabolic, and fixes $[w^\infty]$.
(3) If w is minimal, then the stabilizer of $[w^\infty]$ is generated by g_w.

Proof. Assume that w is right cuspidal, the other case being analogous. Since w is a right-cuspidal cycle, the two sides $[a_1]$ and $[\widehat{a}_r]$ share a common vertex v, say, and v is the left endpoint of $[a_{\widehat{r}}]$. Let $b \in \Sigma$ be such that $v \in [\widehat{a}_r, b]$. Since the arcs $[\widehat{a}_r]$ and $[\widehat{a}_r, b]$ share their left-endpoint, so do the arcs $g_{a_r}[\widehat{a}_r] = \partial\mathbb{D} \smallsetminus [a_r]$ and $g_{a_r}[\widehat{a}_r, b] = [b]$, and therefore the left endpoint $[b]$ coincides with the right endpoint of $[a_r]$. This implies $b = a_r + 1 = \widehat{a}_{r-1}$. We conclude that $v \in [a_1] \cap [\widehat{a}_r, \widehat{a}_{r-1}]$ and therefore in particular $g_{a_r} v \in [a_r] \cap [\widehat{a}_{r-1}]$. Iterating this argument r times, we deduce that $g_w v \in [a_1] \cap [\widehat{a}_r] = \{v\}$, and so $v = g_w v$. Since $v \in C(G)$, we deduce that g_w is parabolic. On the other hand, we also deduce,

$$v \in \bigcap_{\ell \geq 1} \underbrace{[w, \dots, w]}_{\ell \text{ times}} = [w^\infty].$$

This shows the first two points.

Finally, let $\gamma \neq \mathrm{id}$ be in the stabilizer of $v = [w^\infty]$. Since $\{g_a\}_{a\in\Sigma}$ generates G, there exists $m \geq 1$ and $b_1, \dots, b_m \in G$ such that $\gamma = g_{b_1} \cdots g_{b_m}$. We assume that m is minimal, so that for $i = 1, \dots, m-1$ we have $g_{b_i} \neq g_{b_{i+1}}^{-1}$, i.e. $b_i \neq \widehat{b}_{i+1}$. In particular, $w' := b_1 \cdots b_m$ is admissible. We have $v = g_{w'} v \in [w']$ and $v = g_{w'}^{-1} v \in [\widehat{w}']$, so that $v \in [w'] \cap [\widehat{w}']$ and this intersection is non-empty if and only if w' is a cuspidal cycle. Since w is minimal, this implies that either $w' = w^n$ or $w' = \widehat{w}^n$ for some $n \geq 1$, thus proving the claim. □

We now prove an asymptotic estimate for $h = h_w \in \mathcal{H}_c$ when w is a cuspidal cycle.

Lemma 4.5. *Let $w = [a_1, \dots, a_r]$ be a cuspidal cycle. Then,*

(1) For all $x \in \partial\mathbb{D}$ with $\mathrm{dist}(x - [w^\infty]) \gg 1$ we have

$$|x - g_{w^\ell}^{-1}\infty| \gg_w 1.$$

uniformly in $\ell \geq 1$.

(2) There exists a constant $C(w) > 0$ satisfying $C(w) = C(\widehat{w})$ such that for all $x \in \partial\mathbb{D} \smallsetminus [w^\infty]$, we have

$$|g'_{w^\ell}(x)| \sim \frac{C(w)}{\ell^2 d(x, [w^\infty])^2}$$

as $\ell \to \infty$. More precisely, $|g'_{w^\ell}(x)| = \frac{C(w)}{\ell^2 d(x,[w^\infty])^2}(1 + O_w(1/\ell))$ uniformly in x satisfying $\mathrm{dist}(x, [w^\infty]) \gg 1$.

Proof. Write

$$g_{w^\ell} = \begin{pmatrix} \alpha_\ell & \beta_\ell \\ \overline{\beta}_\ell & \overline{\alpha}_\ell \end{pmatrix}, \qquad |\alpha_\ell|^2 - |\beta_\ell|^2 = 1.$$

Since g_w is parabolic, we have $\alpha_\ell + \overline{\alpha}_\ell = \pm 2$ for some choice of sign $\pm$, and then

$$v = \pm 1/\overline{\beta}_\ell - \overline{\alpha}_\ell/\overline{\beta}_\ell. \tag{4.8}$$

However, again since g_w is parabolic, we have $|\beta_\ell| = \ell|\beta_1| \to \infty$. By discreteness of G and since $w \neq \epsilon$ we have $\beta_1 \gg 1$ and thus

$$z_\ell := g_{w^\ell}^{-1}\infty = -\overline{\alpha}_\ell/\overline{\beta}_\ell = v + O(1/\ell).$$

In particular, $z_\ell \to v$. But since $|\alpha_\ell|^2 > |\beta_\ell|^2$, the point z_ℓ lies outside $\overline{\mathbb{D}}$. Let B be a neighborhood of v in $\partial\mathbb{D}$. We deduce that

$$\min_{x\in\partial\mathbb{D}\smallsetminus B} d(x, z_\ell) \gg 1, \tag{4.9}$$

which proves (1). Moreover, for $x \in \partial\mathbb{D} \smallsetminus B$ by (4.8) and (4.9) we have

$$|g'_{w^\ell}(x)| = \frac{1}{\left|\overline{\beta}_\ell x + \overline{\alpha}_\ell\right|^2} = \frac{1}{|\beta_\ell|^2 d(x, z_\ell)^2} = \frac{1}{\ell^2|\beta_1|^2 d(x, v)^2}(1 + O(1/\ell))$$

uniformly in $x \in \partial\mathbb{D} \smallsetminus B$. This yields (2) with $C(w) = |\beta_1|^{-2}$. □

Corollary 4.6. *Let $h = h_w^c \in \mathcal{H}_c$ with w cuspidal. Let w_0 be a minimal cuspidal cycle*[3] *such that $w = w_0^\ell w_1$ for some $\ell \geq 0$ and some prefix $w_1 \neq \epsilon$ of w_0. Then*

$$\|h'\|_\infty \asymp (1 + \ell)^{-2}.$$

[3]Notice that w_0 and ℓ are unique unless $|w| = 1$

Proof. Let $[w_0^\infty] = [a_1 a_2 \cdots]$ and $|w_1| = m \ll 1$. For $x \in \mathrm{Dom}(h_w^c)$, we have
$$(h_w^c)'(x) = g'_{w_0^\ell}(g_{w_1}(x))g'_{w_1}(x).$$
On the one hand, the pole of a matrix in $\mathrm{SU}(1,1)\smallsetminus\{\mathrm{id}\}$ lies outside $\overline{\mathbb{D}}$, so that $\|g'_{w_1}\|_{\infty,\partial\mathbb{D}} < \infty$ for each w_1. Since there are finitely many possibilities for w_0 and thus for w_1, we deduce $|g'_{w_1}(x)| \asymp 1$. Also, $g_{w_1}(\mathrm{Dom}(h_w^c)) = [a_1,\dots,a_{m+1}] \smallsetminus [a_1,\dots,a_{m+2}]$ and thus $\mathrm{dist}(g_{w_1}(\mathrm{Dom}(h_w^c)), [w_0^\infty]) \gg 1$. By Lemma 4.5 we then have $|g'_{w_0^\ell}(g_{w_1}(x))| \asymp 1/\ell^2$ for ℓ sufficiently large. This concludes the proof since the statement is obvious for $\ell \ll 1$. ☐

We also record some results which will be useful in future applications.

Proposition 4.7. *For all $h \in \mathcal{H}_c$ we have*
$$\mathrm{dist}(\mathrm{Im}(h), V) \asymp \|h'\|^{1/2}. \tag{4.10}$$
Moreover, if $h_1 \in \mathcal{H}_c^* \smallsetminus \{\mathrm{id}\}$, $h_2 \in \mathcal{H}_c$ with $\mathrm{Im}(h_2) \subset \mathrm{Dom}(h_1)$, then
$$\mathrm{dist}(\mathrm{Im}(h_2), \partial\mathbb{D} \smallsetminus \mathrm{Dom}(h_1)) \gg \|h_2'\|^{1/2}. \tag{4.11}$$

Proof. Let w be a primitive cuspidal cycle and $\ell \geq 0$. Then, with the same notation as in the proof of Lemma 4.5, for $x \in \partial\mathbb{D}$ with $\mathrm{dist}(x - [w^\infty]) \gg 1$ we have
$$g_{w^\ell}(x) - [w^\infty] = \frac{1}{\overline{\beta}_\ell}\Big(\pm 1 - \frac{1}{\overline{\beta}_\ell x + \overline{\alpha}_\ell}\Big)$$
and thus, since $|\beta_\ell| \asymp \ell$, $|\overline{\beta}_\ell x + \overline{\alpha}_\ell| = |h_w'(x)|^{-1/2} \asymp \ell$, we deduce
$$|g_{w^\ell}(x) - [w^\infty]| \asymp \|h'\|_\infty^{1/2} \tag{4.12}$$
for ℓ large enough, and thus for all $\ell \geq 0$ since (4.12) holds trivially if $\ell \ll 1$.

Now, let w be any cuspidal word. Writing $w = w_0^\ell w_1$ as in Corollary 4.6 and proceeding as in its proof using (4.12) we deduce that
$$|h(x) - [w_0^\infty]| \asymp \|h'\|_\infty^{1/2}.$$
Equation (4.10) then follows since $\mathrm{dist}(\mathrm{Im}(h), V \smallsetminus \{[w_0^\infty]\}) \asymp 1$.

We now prove (4.11). We can assume h_2 is not indexed by one letter word, the result being trivial in that case. The domain $\mathrm{Dom}(h_1)$ is an arc limited on one side by a vertex v_1 and on the other by a point $g_a v_2$ in the interior of $[a]$, for some $a \in \Sigma$ and $v_2 \in V$. By (4.10) we have $\mathrm{dist}(v_1, \mathrm{Im}(h_2)) \ll \|h_2'\|_\infty$. Moreover, since $g_a v_2 \in [a]$ is not a vertex, $\mathrm{dist}(\mathrm{Im}(h_2), g_a v_2) \gg 1$ unless $\mathrm{Im}(h_2) \subset [a]$. Thus, we can assume $h_2 = h_{aw}^c$ for some cuspidal $w \in \Sigma^*$. Then for $x \in \mathrm{Dom}(h_2)$
$$|h_2(x) - g_a v_2| = |g_a g_w x - g_a v_2| \gg |g_w - v_2| \gg \|(h_w^c)'\|_\infty \asymp \|h_2'\|_\infty,$$
where we used (4.10) once again. This concludes the proof. ☐

It will also be convenient to have a version of Lemma 4.1, Corollary 4.2 and Lemma 4.3 for $h \in \mathcal{H}_c$.

Proposition 4.8. *For all $h \in \mathcal{H}_c^* \smallsetminus \{\mathrm{id}\}$ we have*
$$\begin{aligned} &\|h''/h'\|_\infty \ll 1, \\ &|h'(x)| \asymp |h'(0)|, \qquad \forall x \in \mathrm{Dom}(h). \end{aligned} \tag{4.13}$$
Moreover, there exists $\rho \in [0,1)$ such that for all $r \geq 1$ and $h \in \mathcal{H}_c^r$ we have
$$\|h'\|_\infty \ll \rho^r \tag{4.14}$$

Proof. It suffices to prove (4.13) since the remaining claims then follow as for $\mathcal{H}^*$.

Let $h = h^c_{w,r} \in \mathcal{H}^r$ with $w = a_1 \cdots a_n$. We can assume $|w|$ is sufficiently large, since the claim is trivial if $|w| \ll 1$.

Let ψ_w be the indifferent or repelling fixed point of g_w. Then, as in [Sta04, Lemma 4.3][4], we have that $|\psi_w - g_w^{-1}\infty| \to 0$ as $|w| \to \infty$. Since $\frac{h''}{h'}(x) = -2(x - g_w^{-1}\infty)^{-1}$, it is thus sufficient to show that $\mathrm{dist}(\psi_w, \mathrm{Dom}(h)) \gg 1$.

Next, repeating the argument in [Sta04], we notice $g_w[\widehat{w}] = \overline{\partial\mathbb{D} \smallsetminus [a_1]}$ and we claim that $\psi_w \in [\widehat{a}_n]$. Indeed, if $\widehat{a}_n \neq a_1$, then $g_w[\widehat{w}] \supset [\widehat{\omega}]$ and so g_w has a fixed point (which has to be repellent or indifferent) in $[\widehat{w}] \subset [\widehat{a}_n]$, i.e. $\psi_w \in [\widehat{a}_n]$. If instead $\widehat{a}_n = a_1$, then any possible fixed point of g_w in $\partial\mathbb{D} \smallsetminus [a_1] = \partial\mathbb{D} \smallsetminus [\widehat{a}_n]$ would have to be in $[\widehat{w}] \cap (\partial\mathbb{D} \smallsetminus [\widehat{a}_n]) = \varnothing$. Thus, $\psi_w \notin \mathbb{D} \smallsetminus [\widehat{a}_n]$ and so $\psi_w \in [\widehat{a}_n]$.

Finally, recalling the notation $b^\pm := \widehat{b} \pm 1$, we notice that

$$\mathrm{Dom}(h) \cap \big([\widehat{a}_n + 1] \cup [\widehat{a}_n] \cup [\widehat{a}_n - 1]\big) \subset \overline{([a_n^+] \smallsetminus [a_n^+, a_n^{++}]) \cup ([a_n^-] \smallsetminus [a_n^-, a_n^{--}])}.$$

The set on the right is bounded away from $[\widehat{a}_n]$. Since the same also holds for the complementary part of $\mathrm{Dom}(h)$, we get

$$\mathrm{dist}(\psi_w, \mathrm{Dom}(h)) \gg \mathrm{dist}([\widehat{a}_n], \mathrm{Dom}(h)) \gg 1,$$

as desired. ☐

5. Transfer operators

Recall that $\Theta \subset \mathrm{GL}(d, \mathbb{R})$ is a finite subgroup. We denote

$$\partial\mathbb{D}_\Theta := \partial\mathbb{D} \times \Theta, \qquad K_\Theta := K \times \Theta$$

and define $\partial\mathbb{D}'_\Theta, \mathcal{C}\Theta, \dots$ in the same way.

We will be interested in maps

$$f : \partial\mathbb{D}_\Theta \to \mathbb{C},$$

and their orbits under the transfer operator $\mathscr{L}$ associated with $\mathcal{F}_K$. In order to study this transfer operator, we will also need to consider the transfer operator $\mathscr{C}$ associated with $\mathcal{F}_c$.

5.1. Skew-products.

Recall the definition (1.11) of the family of maps (ϑ_g), and the cocycle property (1.12). In the remainder of the paper, we make the following two hypotheses on (ϑ_g).

(1) For each $g \in G$, with $g = g_{a_1} \cdots g_{a_n}$ with $a_1 \cdots a_n \in \Sigma^*$, the map ϑ_g is constant on $[a]$ for all $a \in \Sigma \smallsetminus \{[\widehat{a_n}]\}$.

Regarding the assumption (1), recall from Section 2.2 that the expression $g = g_{a_1} \cdots g_{a_n}$ is unique. Thanks to the assumption (1) we can write, for any $w \in \Sigma^*$ and a with wa admissible,

$$\vartheta_w(a) := \vartheta_{g_w}|_{[a]} \in \Theta. \tag{5.1}$$

(2) There exists $r_0 \in \mathbb{N}$ with the following property. Given any $b_1, b_2 \in \Sigma$ and $\vartheta_0 \in \Theta$, we may find $w \in \Sigma^*$ with $b_1 w b_2$ admissible, such that $\ell_K(b_1 w b_2) = r_0$ and

$$\vartheta_{b_1 w}(b_2) = \vartheta_0. \tag{5.2}$$

[4]We warn the reader that in [Sta04] the roles of g_w and g_w^{-1} are swapped.

Note that with the notation (5.1), the cocycle property implies

$$\vartheta_{a_1\cdots a_n b_1\cdots b_m}(c) = \vartheta_{b_1\cdots b_m}(c)\vartheta_{a_1\ldots a_n}(b_1). \tag{5.3}$$

We also note, for potential use in future applications, that assumption (1) is invoked in our work only for $[a] \cap \mathrm{Dom}(h) \neq \varnothing$ where $h = h_g$ is an inverse branch in $\mathcal{H}$ or $\mathcal{H}_c$.

We define a map $\mathcal{F}_\Theta : \mathcal{D}_\Theta \to K_\Theta$ by setting

$$\mathcal{F}_\Theta(x, \vartheta_0) := (\mathcal{F}_K(x), \vartheta_0\vartheta_g(x)), \qquad (\mathcal{F}_K(x) = gx, g \in G).$$

Using the cocycle rule (1.12), the inverse branches of $\mathcal{F}_\Theta$ are given by

$$\{(y, \vartheta_1) \mapsto (h(y), \vartheta_1\vartheta_h(y)),\ h \in \mathcal{H}\},$$

where for $h = h_w$ in $\mathcal{H}$, $\mathcal{H}_c$ or $\mathcal{H}_c^\sharp$,

$$\vartheta_h(y) := \vartheta_{g_w}(y) = \vartheta_{g_w^{-1}}(h(y))^{-1}.$$

5.2. **The s-twisted transfer operator.** Define, for all $s \in \mathbb{C}$, the s-twisted transfer operator acting on a map $f : \partial\mathbb{D}_\Theta \to \mathbb{C}$ by

$$\begin{aligned} \mathscr{L}_s[f] : \quad K_\Theta &\to \mathbb{C}, \\ (y, \vartheta_1) &\mapsto \sum_{\substack{h \in \mathcal{H} \\ y \in \mathrm{Dom}(h)}} |h'(y)|^s f(h(y), \vartheta_1\vartheta_h(y)) \end{aligned} \tag{5.4}$$

whenever the sum converges absolutely.

We will also need to define analogous operators for the map $\mathcal{F}_c$. We will give two versions, one with the inverse branches (3.10) in $\mathcal{H}_c$ and one with its restricted version $\mathcal{H}_c^\sharp$ defined in (3.11). Given $f : \partial\mathbb{D}_\Theta \to \mathbb{C}$ as above, we let

$$\mathscr{C}_s[f](y, \vartheta_1) = \sum_{\substack{h \in \mathcal{H}_c \\ y \in \mathrm{Dom}(h)}} |h'(y)|^s f(h(y), \vartheta_1\vartheta_h(y)), \qquad (y, \vartheta_1) \in \partial\mathbb{D}_\Theta$$

and let $\mathscr{C}_s^\sharp[f](y, \vartheta_1)$ be defined in the same way but where the sum ranges over $h \in \mathcal{H}_c^\sharp$. By definition this means that

$$\mathscr{C}_s^\sharp[f](y, \vartheta_1) = \begin{cases} \mathscr{C}_s[f](y, \vartheta_1), & (y \notin K), \\ 0 & (y \in K). \end{cases}$$

Note that by the triangle inequality, for $\sigma, \tau \in \mathbb{R}$ we have

$$|\mathscr{C}_{\sigma+i\tau}[f]| \leq \mathscr{C}_\sigma[|f|]$$

whenever the right-hand side is finite.

We also define the untwisted versions of $\mathscr{L}$, $\mathscr{C}_s$ and $\mathscr{C}_s^\sharp$ by

$$\mathscr{L} := \mathscr{L}_1, \qquad \mathscr{C} := \mathscr{C}_1, \qquad \mathscr{C}^\sharp := \mathscr{C}_1^\sharp.$$

Notice that for a non-negative f we have $\mathscr{C}^\sharp[f] \leq \mathscr{C}[f]$.

Recall the definition of φ from (3.19). By Proposition 3.7, we have

$$\mathscr{C}[\varphi] = \varphi$$

and thus, since $\varphi \asymp 1$, for bounded f we have $\mathscr{C}^\sharp[f], \mathscr{C}[f] \ll 1$. Finally, for all f integrable on $\partial\mathbb{D}$, we have

$$\int_{\partial\mathbb{D}} \mathscr{C}[f](y)\,\mathrm{d}s(y) = \int_{\partial\mathbb{D}} f(y)\,\mathrm{d}s(y). \tag{5.5}$$

This can also be stated as the fact that the dual $\mathscr{C}^*$ of $\mathscr{C}$ preserves the Lebesgue measure.

The relevance of $\mathscr{C}_s$ and $\mathscr{C}_s^\sharp$ is evident from Lemma 3.3. Indeed, one immediately deduces that for $y \in K$ and $\vartheta_1 \in \Theta$, we have

$$\begin{aligned} \mathscr{L}_s[f](y,\vartheta_1) &= \sum_{m=0}^{\infty} \mathscr{C}_s(\mathscr{C}_s^\sharp)^m[f](y,\vartheta_1) \\ &= \mathscr{C}_s(\mathrm{Id} - \mathscr{C}_s^\sharp)^{-1}[f](y,\vartheta_1), \end{aligned} \tag{5.6}$$

whenever the series defining $\mathscr{L}_s[f]$ is absolutely convergent.

5.3. **Convergence for $\mathfrak{Re}(s)$ below 1.** Our next goal is to show that there exists $\sigma_0 < 1$ such that the sum in the definition (5.4) of $\mathscr{L}_s[f]$ converges normally for all bounded f and all $s \in \mathbb{C}$ with $\mathfrak{Re}(s) \geq \sigma_0$. We start with the following lemma which is relative to the case when $\mathfrak{Re}(s) = 1$.

Lemma 5.1. *We have $(\mathscr{C}^\sharp)^2[\varphi] < \rho\varphi$ for some $\rho \in (0,1)$.*

Proof. By compactness of $\partial\mathbb{D}$, it suffices to show the strict inequality with $\rho = 1$. We have $0 = \mathscr{C}^\sharp[\varphi](y) < \mathscr{C}[\varphi](y) = \varphi(y)$ for all $y \in K$. If $y \notin K$ the two sums coincides and one has $\mathscr{C}^\sharp[\varphi](y) = \mathscr{C}[\varphi](y) = \varphi(y)$. In particular, letting $f := \mathscr{C}^\sharp[\varphi]$, we have $0 \leq f \leq \varphi$ and $f < \varphi$ on K. If $y \in K$ then $\mathscr{C}^\sharp[f](y) = 0 < \mathscr{C}[\varphi](y)$, as desired. On the other hand if $y \notin K$, $y = [b, \dots]$, then we can always find $h \in \mathcal{H}_c$ with $y \in \mathrm{Dom}(h)$ and $h(y) \in K$; it suffices to pick $h = h_a^c$ for some $a \in \Sigma$ satisfying $[a,b] \subset K$, where we used the notation (3.9). For this h we have $f(h(y)) < \varphi(h(y))$ and therefore $\mathscr{C}^\sharp[f](y) < \mathscr{C}[\varphi](y)$ as well. ☐

Lemma 5.2. *Let $\varepsilon > 0$. Then, for all $\sigma > 1/2 + \varepsilon$ we have $\mathscr{C}_\sigma[\mathbf{1}] \ll_\varepsilon \mathbf{1}$.*

Proof. This follows from Corollary 4.6 since there are only finitely possibilities for w_0 and w_1. ☐

We deduce the following contraction result for $(\mathscr{C}^\sharp)^2$ on the left of $\mathfrak{Re}(s) = 1$ line.

Lemma 5.3. *There exists $\sigma_0 < 1$ and $\rho < 1$ such that for $\sigma_0 \leq \sigma \leq 1$*

$$(\mathscr{C}_\sigma^\sharp)^2[\varphi] < \rho\varphi.$$

Proof. The result follows by continuity combining Lemma 5.1 and 5.2. ☐

Proposition 5.4. *Let $f : \partial\mathbb{D}_\Theta \to \mathbb{C}$ be bounded. There exists $\sigma_0 \in (0,1)$ such that the sum in* (5.4) *converges absolutely and uniformly for all $s \in \mathbb{C}$ with $\mathfrak{Re}(s) \geq \sigma_0$.*

Proof. Take σ_0 be as in Lemma 5.3. Then for all $m \geq 1$ we have $(\mathscr{C}_{\sigma_0}^\sharp)^m[f] \ll (\mathscr{C}_{\sigma_0}^\sharp)^m[\varphi] \leq \rho^m\varphi$, with an implicit constant independent of m. The result then follows by (5.6). ☐

5.4. **Spectral properties at $s = 1$.** Finally we record the properties of $\mathscr{L}$. By Proposition 5.4, we know that $\mathscr{L}$ is a bounded operator from $L^\infty(\partial\mathbb{D}_\Theta)$ to $L^\infty(K_\Theta)$. Recall the definition of ξ in (3.17). We extend ξ to a map on $\partial\mathbb{D}_\Theta$ supported inside K_Θ by letting it be constant on the second coordinate:

$$\xi(x,\vartheta_0) := \begin{cases} \xi(x), & (x,\vartheta_0) \in K_\Theta, \\ 0, & \text{otherwise.} \end{cases}$$

By (3.17) we have that ξ is an eigenfunction of eigenvalue 1 for $\mathscr{L}$,

$$\mathscr{L}[\xi] = \xi. \tag{5.7}$$

By Proposition 3.6, the absolutely continuous measure $\xi(x)\,\mathrm{d}s(x)$ is $\mathcal{F}_K$-invariant, and analogously to (5.5), we also have

$$\int_{K_\Theta} \mathscr{L}[f](y,\vartheta_1)\,\mathrm{d}s(y)\,\mathrm{d}\vartheta_1 = \int_{\partial\mathbb{D}_\Theta} f(y,\vartheta_1)\,\mathrm{d}s(y)\,\mathrm{d}\vartheta_1 \tag{5.8}$$

for all f integrable on $\partial\mathbb{D}_\Theta$, where $\mathrm{d}\vartheta_1$ denotes the uniform counting probability measure on Θ.

5.5. **Cost functions.** We let $\boldsymbol{\phi} = (\boldsymbol{\phi}_h)_{h\in\mathcal{H}}$ be a family of maps $\boldsymbol{\phi}_h : \mathrm{Dom}(h) \to \mathbb{R}^d$ which belong to H^{κ_0} for some $\kappa_0 \in (0,1]$, where we recall H^κ is introduced in Section 1.5. Since we have the disjoint union $\partial\mathbb{D}' = \bigcup_{h\in\mathcal{H}} \mathrm{Im}(h)$, the family $(\boldsymbol{\phi}_h)_{h\in\mathcal{H}}$ can be equivalently reinterpreted as a function Φ_K on $\partial\mathbb{D}'_\Theta$ defined as

$$\Phi_K(x,\vartheta_0) := \vartheta_0\vartheta_h(y)^{-1}\boldsymbol{\phi}_h(y) \qquad\qquad x = h(y) \text{ for some } h \in \mathcal{H}, \tag{5.9}$$

$$\Phi_K(x) := \Phi_K(x,1). \tag{5.10}$$

By construction we then also have

$$\vartheta_1\boldsymbol{\phi}_h(y) = \Phi_K(h(y),\vartheta_1\vartheta_h(y)) \quad \forall h\in\mathcal{H},\ y\in\mathrm{Dom}(h). \tag{5.11}$$

It will be useful to define $\boldsymbol{\phi}_h$ also for inverse branches of any depth, by letting $\boldsymbol{\phi}_h = 0$ if $h = \mathrm{id}$ and otherwise

$$\boldsymbol{\phi}_h(y) = \sum_{i=1}^n \vartheta_{h_{i+1}\circ\cdots\circ h_n}(y)\boldsymbol{\phi}_{h_i}(h_{i+1}\circ\cdots\circ h_n(y)) \tag{5.12}$$

for $h = h_1\circ\cdots\circ h_n \in \mathcal{H}^n$, $h_i \in \mathcal{H}$.

Remark 5.5. In the introduction, we discussed for simplicity a situation where the family of maps $(\boldsymbol{\phi}_g)$ is defined for all $g \in G$. In such a situation, the relation 5.12 holds automatically by the cocycle property (1.14).

5.6. **The transfer operator $\mathscr{L}_{s,\boldsymbol{t}}$.** We can now define the main generating function we will work with. For all $s \in \mathbb{C}$ with $\mathfrak{Re}(s) \geq \sigma_0$ the quantity defined in Proposition 5.4, all $\boldsymbol{t} \in \mathbb{R}^d$ and all bounded map $f : \partial\mathbb{D}_\Theta \to \mathbb{C}$, we let

$$\begin{array}{rccl} \mathscr{L}_{s,\boldsymbol{t}}[f]: & K_\Theta & \to & \mathbb{C}, \\ & (y,\vartheta_1) & \mapsto & \displaystyle\sum_{\substack{h\in\mathcal{H}\\ y\in\mathrm{Dom}(h)}} |h'(y)|^s \mathrm{e}(\langle \boldsymbol{t}, \vartheta_1\boldsymbol{\phi}_h(y)\rangle) f(h(y),\vartheta_1\vartheta_h(y)). \end{array} \tag{5.13}$$

Remark 5.6. Any bounded function $f : K_\Theta \to \mathbb{C}$ extends trivially to a function on $\partial\mathbb{D}_\Theta$ by setting it to zero on $\partial\mathbb{D}_\Theta \smallsetminus K_\Theta$. With a slight abuse of notation, we also denote $\mathscr{L}_{s,\boldsymbol{t}}$ the operator acting on bounded functions on K_Θ, obtained by applying the original operator (5.13) to this extension. Clearly, in this case (5.8) holds also with $\partial\mathbb{D}_\Theta$ in the right-hand side replaced by K_Θ.

6. Moments conditions

In this section we shall introduce some hypothesis on the period functions $(\boldsymbol{\phi}_h)$ which we will need in order to carry out the functional analysis of the operators $\mathscr{L}_{s,\boldsymbol{t}}$.

Given $\kappa \in (0,1]$, we recall the definition of κ-Hölder norm and semi-norm given in Section 1.5. We also quote from [BD22, Section 4.1] the following properties of $\|\|_{(\kappa)}$:

$$\|fg\|_{(\kappa)} \leq \|f\|_{(\kappa)}\|g\|_\infty + \|g\|_{(\kappa)}\|f\|_\infty, \qquad f \in H^\kappa, \tag{6.1}$$

$$\|f\circ h\|_{(\kappa),\mathrm{Dom}(h)} \ll \|h'\|_\infty \|f\|_{(\kappa),\mathrm{Im}(h)}, \qquad f \in H^\kappa, h \in \mathcal{H}^*, \tag{6.2}$$

$$\|e^{if}\|_{(\kappa)} \ll \|f\|_{(\kappa')}^{\kappa/\kappa'} \qquad f \in H^{\kappa'}, \kappa' \in [\kappa, 1], \tag{6.3}$$

$$\|f\|_{(\kappa)} \leq \|f\|_\infty^{1-\kappa/\kappa'}\|f\|_{(\kappa')}^{\kappa/\kappa'} \qquad f \in H^{\kappa'}, \kappa' \geq \kappa. \tag{6.4}$$

Hypothesis (M_K) (Moments conditions)**.** *For given positive numbers $\alpha, \kappa_0, \lambda$, we have*

$$\sum_{h\in\mathcal{H}} \|h'\|_\infty \|\boldsymbol{\phi}_h\|_{\infty,\mathrm{Dom}(h)}^\alpha < \infty, \tag{6.5}$$

$$\sum_{h\in\mathcal{H}} \|h'\|_\infty \|\boldsymbol{\phi}_h\|_{(\kappa_0),\mathrm{Dom}(h)}^\lambda < \infty. \tag{6.6}$$

Note that the sums are over $\mathcal{H}$, the inverse branches of $\mathcal{F}_K$, see (3.3). For our applications in [BDLa, BDLb], it will be crucial to reduce the sum over h to a smaller set. This is the point of the following definition.

Hypothesis (M_c) (Cuspidal moments conditions)**.** *For given positive numbers $\alpha, \kappa_0, \lambda$ there exist a family $(\boldsymbol{\phi}_h)_{h\in\mathcal{H}_c}$ of κ_0-Hölder continuous maps from* $\mathrm{Dom}(h)$ *to* $\mathbb{R}^d$ *such that the following conditions are satisfied.*

(1) For all $(x,\vartheta) \in \partial\mathbb{D}'_\Theta$, we have

$$\Phi_K(x,\vartheta_0) = \vartheta_0 \sum_{j=0}^{r_x-1} \Phi_c(\mathcal{F}_c^j(x)), \tag{6.7}$$

with r_x as in (3.13) *and $\Phi_c : \partial\mathbb{D}' \to \mathbb{R}^d$ defined by*[5]

$$\Phi_c(x) := \vartheta_h(y)^{-1}\boldsymbol{\phi}_h(y), \qquad x = h(y) \text{ for some } h \in \mathcal{H}_c, \tag{6.8}$$

which is well-posed since $\partial\mathbb{D}' = \bigcup_{h\in\mathcal{H}_c} \mathrm{Im}(h)$, where the union is disjoint.

(2) We have

$$\sum_{h_1,h_2\in\mathcal{H}_c} \|h_1'\|_\infty \|h_2'\|_\infty \|\boldsymbol{\phi}_{h_1}|_{\mathrm{Im}(h_2)\cap\mathrm{Dom}(h_1)}\|_\infty^\alpha < \infty, \tag{6.9}$$

$$\sum_{h_1,h_2\in\mathcal{H}_c} \|h_1'\|_\infty \|h_2'\|_\infty \|\boldsymbol{\phi}_{h_1}|_{\mathrm{Im}(h_2)\cap\mathrm{Dom}(h_1)}\|_{(\kappa_0)}^\lambda < \infty. \tag{6.10}$$

We remark that condition (6.7) is natural when, as in our applications, one has functions $(\boldsymbol{\phi}_g)_g$ for each $g \in G$ which satisfy the cocycle condition

$$\boldsymbol{\phi}_{g_1g_2}(x) = \boldsymbol{\phi}_{g_2}(x) + \vartheta_{g_2}(x)\boldsymbol{\phi}_{g_1}(g_2x) \qquad \forall g_1, g_2 \in G.$$

Indeed, in this case, (6.7) amounts to say that the cocycle relation holds for the decompositions (3.12). We extend the notation $\boldsymbol{\phi}_h$ to $h \in \mathcal{H}_c^n$ for any $n \geq 0$ in the same way as (5.12).

Assuming Hypothesis (M_c), for all $s \in \mathbb{C}$ with $\mathfrak{Re}(s) > 1/2$ and all $\boldsymbol{t} \in \mathbb{R}^d$, we define the $(s,\boldsymbol{t})$-twisted operators $\mathscr{C}_{s,\boldsymbol{t}}, \mathscr{C}_{s,\boldsymbol{t}}^\sharp$ acting on maps $f : \partial\mathbb{D}_\Theta \to \mathbb{C}$ by

$$\mathscr{C}_{s,\boldsymbol{t}}[f](y,\vartheta_1) = \sum_{\substack{h\in\mathcal{H}_c\\ y\in\mathrm{Dom}(h)}} |h'(y)|^s \mathrm{e}(\langle \boldsymbol{t}, \vartheta_1\boldsymbol{\phi}_h(y)\rangle) f(h(y), \vartheta_1\vartheta_h(y))$$

[5]Equivalently, $\vartheta_h\boldsymbol{\phi}_h(x) = \Phi_c(h(x))$, $\forall h \in \mathcal{H}_c,\ x \in \mathrm{Dom}(h)$.

and $\mathscr{C}^\sharp_{s,\boldsymbol{t}}$ defined in the same way with $\mathcal{H}_c^\sharp$ instead of $\mathcal{H}_c$. Notice that (6.7) implies that the following analogue of (5.6) holds:

$$\mathscr{L}_{s,\boldsymbol{t}}[f](y,\vartheta_1) = \mathscr{C}_{s,\boldsymbol{t}}(\mathrm{Id} - \mathscr{C}^\sharp_{s,\boldsymbol{t}})^{-1}[f](y,\vartheta_1), \qquad (y,\vartheta_1) \in K_\Theta \tag{6.11}$$

for $\mathfrak{Re}(s) \geq \sigma_0$.

The advantage of Hypothesis (M_c) over Hypothesis (M_K) lies in the fact that the sums over h_i in (6.9)–(6.10) are restricted to the set $\mathcal{H}_c$, which is easier to parametrize since it consists essentially of powers of finitely many parabolic elements. On the other hand, the domains of $\boldsymbol{\phi}_{h_1}$ in conditions (6.9) and (6.10) are now larger and in particular can contain the vertices of the fundamental domain which pose additional challenges in applications.

Lemma 6.1 ((M_c) implies (M_K))**.** *Let $\alpha_1 > 0, \lambda_1 > 0$ and $\kappa_0 > 0$. Suppose that (M_c) holds in the range of values $\alpha < \alpha_1$ and $\lambda < \lambda_1$. Then (M_K) holds in the same range.*

Proof. We recall the definition (3.13) of the return time r_x of $\mathcal{F}_c$ to K and its relation (3.14) with the decomposition of h. Then, from Lemma 3.3 and (3.13) we deduce that for all $\ell \in \mathbb{N}$, we have

$$\sum_{\substack{h\in\mathcal{H} \\ y\in\mathrm{Dom}(h)}} (r_{h(y)})^\ell |h'(y)| = \sum_{r\geq 1} r^\ell \mathscr{C}(\mathscr{C}^\sharp)^{r-1}[\mathbf{1}](y) \ll_\ell 1 \tag{6.12}$$

uniformly in $y \in \partial\mathbb{D}$, by Lemma 5.1. Next, if h decomposes as $h = h_1 \circ \cdots \circ h_{r_h}|_{\mathrm{Dom}(h)}$ as in (3.12), then for all $y \in K$ and all $\varepsilon > 0$, by the cocycle property (6.7), we have

$$\begin{aligned} |\boldsymbol{\phi}_h(y)|^\alpha &\leq \Big(\sum_{j=1}^{r_h} \big|\boldsymbol{\phi}_{h_j}(h_{j+1}\circ\cdots\circ h_{r_h}(y))\big|\Big)^\alpha \leq \sum_{j=1}^{r_h} r_h^\alpha \big|\boldsymbol{\phi}_{h_j}(h_{j+1}\circ\cdots\circ h_k(y))\big|^\alpha \\ &\leq r_h^{1+\alpha+\alpha/\varepsilon} + \sum_{j=1}^{r_h} \big|\boldsymbol{\phi}_{h_j}(h_{j+1}\circ\cdots\circ h_k(y))\big|^{\alpha(1+\varepsilon)} \end{aligned}$$

by Young's inequality. Now, let $\alpha < \alpha_1$ and let $\varepsilon > 0$ be small enough that $\alpha_2 := \alpha(1+\varepsilon) < \alpha_1$. Moreover, let

$$\widetilde{\mathscr{C}}[f](y) := \sum_{\substack{h\in\mathcal{H}_c \\ y\in\mathrm{Dom}(h)}} |h'(y)||\boldsymbol{\phi}_h(y)|^{\alpha_2},$$

and similarly let $\widetilde{\mathscr{C}}^\sharp$ be defined by the same expression restricted to $h \in \mathcal{H}_c^\sharp$. We claim that

$$\widetilde{\mathscr{C}}[1](y) \ll 1, \qquad (y \in K), \tag{6.13}$$

$$\mathscr{C}\widetilde{\mathscr{C}}[1](y) \ll 1, \qquad (y \in \partial\mathbb{D}). \tag{6.14}$$

Indeed, whenever $y \in K$, say $y \in K \cap [a]$ for some $a \in \Sigma$, by definition of $\mathcal{F}_c$ (3.7), we have $y = h_2(y_2)$ with $h_2 = g_a^{-1}$ and some $y_2 \in \mathrm{Dom}(h_2)$. Since a varies in a finite set, we get $|h_2'(y_2)| \asymp 1$, and so

$$\widetilde{\mathscr{C}}[1](y) = \sum_{h_1\in\mathcal{H}} |h_1'||\boldsymbol{\phi}_{h_1}(h_2(y_2))|^{\alpha_2} \ll 1$$

by (6.9) and positivity. This shows the first item. To prove the second item, we write

$$\mathscr{C}\widetilde{\mathscr{C}}[1](y) = \sum_{\substack{h_1,h_2\in\mathcal{H}_c \\ y\in\mathrm{Dom}(h_2)}} |h_1'(h_2(y))h_2'(y)||\boldsymbol{\phi}_{h_1}(h_2(y))|^{\alpha_2} \ll 1$$

by (6.9) again.

By unfolding the decomposition (3.12), we have for all $y \in K$

(6.15)

$$\begin{aligned}\sum_{\substack{h\in\mathcal{H}\\ y\in\mathrm{Dom}(h)}} |h'(y)| \sum_{j=1}^{r_h} |\boldsymbol{\phi}_{h_j}(h_{j+1}\circ\cdots\circ h_{r_h}(y))|^{\alpha_2} &= \big[\widetilde{\mathscr{C}} + \mathscr{C}(\mathrm{Id}-\mathscr{C}^\sharp)^{-1}\widetilde{\mathscr{C}^\sharp}\big](\mathrm{Id}-\mathscr{C}^\sharp)^{-1}[\mathbf{1}](y)\\ &\ll \big[\widetilde{\mathscr{C}} + \mathscr{C}\widetilde{\mathscr{C}^\sharp} + \mathscr{C}(\mathrm{Id}-\mathscr{C}^\sharp)^{-1}\mathscr{C}^\sharp\widetilde{\mathscr{C}^\sharp}\big][\mathbf{1}](y),\end{aligned}$$

where we bounded the rightmost term $(\mathrm{Id}-\mathscr{C}^\sharp)^{-1}[\mathbf{1}]$ using the spectral radius inequality $\mathrm{spr}(\mathscr{C}^\sharp) < 1$ implied by Lemma 5.1. We have $\widetilde{\mathscr{C}}[\mathbf{1}](y) \ll 1$ for all $y \in K$ by (6.13), and $\mathscr{C}^\sharp\widetilde{\mathscr{C}^\sharp}[\mathbf{1}](y) \le \mathscr{C}\widetilde{\mathscr{C}^\sharp}[\mathbf{1}](y) \le \mathscr{C}\widetilde{\mathscr{C}}[\mathbf{1}](y) \ll 1$ for all $y \in \partial\mathbb{D}$ by (6.14), and so the above is uniformly bounded, and (6.5) follows

We now turn to (6.6). With the same notation and the same argument as above we have

$$\begin{aligned}\|\boldsymbol{\phi}_h\|^{\lambda}_{(\kappa_0),\mathrm{Dom}(h)} &= \Big\|\sum_{j=1}^{r_h} \vartheta_{h_j}\boldsymbol{\phi}_{h_j}\circ h_{j+1}\circ\cdots\circ h_{r_h}\Big\|^{\lambda}_{(\kappa_0),\mathrm{Dom}(h_{r_h})}\\ &\le r_h^{1+\lambda+\lambda/\varepsilon} + \sum_{j=1}^{r_h}\|\boldsymbol{\phi}_{h_j}\circ h_{j+1}\circ\cdots\circ h_{r_h}\|^{\lambda(1+\varepsilon)}_{(\kappa_0)\,\mathrm{Dom}(h_{r_h})}.\end{aligned}$$

By (6.2) and (4.14) we have

$$\begin{aligned}\|\boldsymbol{\phi}_{h_j}\circ h_{j+1}\circ\cdots\circ h_{r_h}\|_{(\kappa_0),\mathrm{Dom}(h_{r_h})} &\le \|(h_{j+1}\circ\cdots\circ h_{r_h})'\|^{\kappa_0}_{\infty,\mathrm{Dom}(h_{r_h})}\cdot\|\boldsymbol{\phi}_{h_j}\|_{(\kappa_0),\mathrm{Dom}(h_j)\cap\mathrm{Im}(h_{j+1})}\\ &\ll \|\boldsymbol{\phi}_{h_j}\|_{(\kappa_0),\mathrm{Dom}(h_j)\cap\mathrm{Im}(h_{j+1})}\end{aligned}$$

where by convention $h_{r_h+1} \in \mathcal{H}_c$ denotes any of the finite element such that $\mathrm{Im}(h_{r_h+1}) \subset \mathrm{Dom}(h)$ (note that $\mathrm{Dom}(h)$ is covered by such sets). Thus, by (6.12),

$$\sum_{h\in\mathcal{H}} |h'|\|\boldsymbol{\phi}_h\|^{\lambda}_{(\kappa_0),\mathrm{Dom}(h)} \ll_\varepsilon 1 + \sum_{h\in\mathcal{H}}|h'|\sum_{j=1}^{r_h}\|\boldsymbol{\phi}_{h_j}\|^{\lambda(1+\varepsilon)}_{(\kappa_0),\mathrm{Dom}(h_{r_j})\cap\mathrm{Im}(h_{r_{j+1}})}$$

and we conclude by the same arguments as in (6.15). □

For convenience we record the following variant of (6.5), which comes at almost no cost.

Lemma 6.2. *Let $\alpha_1 > 0$, and assume that the bound* (6.5) *holds for all $\alpha < \alpha_1$. Then for any $m \ge 1$ and any $\alpha < \alpha_1$, we have*

$$\sum_{h\in\mathcal{H}} \|h'\cdot(\log|h'|)^m\cdot|\boldsymbol{\phi}_h|^\alpha\|_{\infty,\mathrm{Dom}(h)} < \infty.$$

Proof. The sum $\sum_{h\in\mathcal{H},y\in\mathrm{Dom}(h)}|h'(y)|\log(|h'(y)|)^M$ is finite for all $M \ge 1$ by Proposition 5.4. The claimed bound follows from Hölder's inequality along with (6.5). □

7. Spectral analysis of transfer operators

In this section we set up the functional spaces on which $\mathscr{L}_{s,\boldsymbol{t}}$ will act and show uniform spectral gap and Dolgopyat-type estimate, generalizing the arguments in [BV05b, BD22, LS25]. These will be the main ingredients to establish Theorem 1.3.

From this section onwards, we shall assume Hypothesis (M_K) holds. For certain results, we shall also require the stronger Hypothesis (M_c), which will be assumed explicitly.

7.1. **Functional space.** Given $0 < \kappa \leq \kappa_0$ and $(s, \boldsymbol{t}) \in \mathbb{C} \times \mathbb{R}^d$ with $\mathfrak{Re}(s) \geq \sigma_0$, we will at first consider $\mathscr{L}_{s,\boldsymbol{t}}$ as an operator acting on H^κ (cf. Remark 5.6). Notice that $\mathscr{L}_{s,\boldsymbol{t}}[f]$ is indeed in H^κ whenever $f \in H^\kappa$ since the indicator function for the condition $y \in \mathrm{Dom}(h)$ in (5.13) is locally constant on K.

Throughout the rest of the paper, we fix an arbitrary $\varepsilon \in (0, \alpha/2)$, on which all subsequent implicit bounds might depend, even if not explicitly indicated. We also fix a value $\kappa > 0$ satisfying

$$\kappa \leq \frac{\kappa_0 \min(\lambda, \alpha/2, 1)(\alpha - \min(1, \alpha - \varepsilon))}{2(\alpha - \min(\lambda, \alpha/2, 1))}. \tag{7.1}$$

Notice that this in particular implies

$$\kappa \leq \frac{\min(1, \lambda)}{2} \kappa_0, \tag{7.2}$$

which will be used in later estimates (see e.g. the proof of Lemma 7.6), and that (7.1) simplifies to $\kappa \leq \kappa_0/2$ if $\alpha \geq 2$ and $\lambda \geq 1$.

Remark 7.1. We take the opportunity to mention that the choice of κ in [BD22, eq. (5.3)] should be corrected similarly to (7.1). This is needed to prove the analogue of (7.7) which is implicitly used in Section 7 therein.

Lemma 7.2. *The space $(H^\kappa, \|\cdot\|_\kappa)$ is a Banach space, and the norm $\|\cdot\|_\kappa$ is pre-compact with respect to $\|\cdot\|_\infty$.*

Proof. The first claim follows from the bijection $H^\kappa \to \bigoplus_{a \in \Sigma} H^\kappa([a])$ by finiteness of Σ, and the second claim is implied by Arzela–Ascoli's theorem. ☐

7.2. **Dominant spectral properties.** The goal of this subsection is to prove the following Ruelle–Perron–Frobenius Theorem (see [Bal00, Theorem 1.5]) for the family $\mathscr{L}_{\sigma,0}$ with σ real. Recall that σ_0 is defined in Lemma 5.3 and that spr denotes the spectral radius.

Proposition 7.3. *For all $\sigma \in \mathbb{R}$ with $\sigma \geq \sigma_0$, the following hold.*

(i) The operator $\mathscr{L}_{\sigma,0} : H^\kappa \to H^\kappa$ is quasi-compact. It has a maximal eigenvalue $\lambda_{\sigma,0} > 0$ with a corresponding positive eigenfunction $\xi_{\sigma,0}$.

(ii) The dominant eigenvalue $\lambda_{\sigma,0}$ is simple and there is a centered disk of radius strictly less than $\lambda_{\sigma,0}$ which contains all other eigenvalues.

(iii) In particular, $\mathscr{L}$ has leading eigenvalue 1 with eigenfunction ξ and its spectral decomposition is given by

$$\mathscr{L} = \mathscr{P} + \mathscr{N}$$

where $\mathscr{P}$ is a projection, $\mathrm{spr}(\mathscr{N}) < 1$, and $\mathscr{P}\mathscr{N} = \mathscr{N}\mathscr{P} = 0$.

This shows alternatively the existence of an invariant measure. Note that $\mathscr{L}\xi = \xi$ is equivalent to that there is a unique probability measure ν such that $\mathscr{L}^*\nu = \nu$ and $\int \xi d\nu = 1$ by means of the dual operator. By (5.8) we have

$$\mathrm{d}\nu(x, v) = \mathrm{d}s(x)\,\mathrm{d}v.$$

The projector in the spectral decomposition has the explicit formula:

$$\mathscr{P} f = \Big(\int_{K_\Theta} f \,\mathrm{d}\nu \Big) \xi. \tag{7.3}$$

Proposition 7.3 is proven by following the arguments in [Bal00, Theorem 1.5], and [LS25, Theorem 5.9] for skew-product extensions. We highlight the two most important ingredients in the proof.

Lemma 7.4. *Let ρ be as in Lemma 4.3. There exists a function $(\sigma_0, \infty) \ni \sigma \mapsto A_\sigma \in \mathbb{R}_+$ which is continuous, piecewise smooth, decreasing, and with $A_1 = 1$ such that for all $r \geq 1$ and $\sigma_0 \leq \sigma \ll 1$,*

$$\|\mathscr{L}^r_{\sigma,0} f\|_\infty \ll A^r_\sigma \|f\|_\infty, \tag{7.4}$$

$$\|\mathscr{L}^r_{\sigma,0} f\|_{(\kappa)} \ll A^r_\sigma (\|f\|_\infty + \rho^{\kappa r}\|f\|_{(\kappa)}), \tag{7.5}$$

where the implicit constant depends at most the choice of the fundamental domain R and the boundedness of σ.

Proof. By Lemma 4.3, there exists $m \geq 1$ such that $\|h'(x)\|_\infty < 1$ for all $h \in \mathcal{H}^m$. For $\sigma \geq \sigma_0$, let $A_\sigma := \left\|\mathscr{L}^m_{\sigma,0}[\xi]/\xi\right\|^{1/m}_\infty$. In particular, $A_1 = 1$ by (5.7). It is also clearly continuous, piecewise smooth and decreasing. Moreover, for $f \in H^\kappa$, we have

$$\|\mathscr{L}^m_{\sigma,0}[f]/\xi\|_\infty = \left\|\mathscr{L}^m_{\sigma,0}\left[\tfrac{f}{\xi}\xi\right]/\xi\right\|_\infty \leq \|f/\xi\|_\infty \left\|\mathscr{L}^m_{\sigma,0}[\xi]/\xi\right\|_\infty = A^m_\sigma \|f/\xi\|_\infty,$$

by positivity. By induction, $\|\mathscr{L}^{rm}_{\sigma,0}[f]/\xi\|_\infty \leq A^{rm}_\sigma \|f/\xi\|_\infty$ for $r \geq 1$ and (7.4) follows since $\xi \asymp 1$ and $\|\mathscr{L}_{\sigma,0}\|_\infty \ll 1$ for $\sigma_0 < \sigma \ll 1$. Regarding the second estimate, for all $r \geq 1$ we have

$$\begin{aligned}\|\mathscr{L}^r_{\sigma,0} f\|_{(\kappa)} &\leq \sum_{h \in \mathcal{H}^r} \||h'|^\sigma \cdot f \circ h\|_{(\kappa),\mathrm{Dom}(h)} \\ &\ll \sum_{h \in \mathcal{H}^r} \left(\||h'|^\sigma\|_{(\kappa)} \|f\|_\infty + \|h'\|^{\sigma+\kappa}_\infty \|f\|_{(\kappa)}\right)\end{aligned}$$

by (6.1)-(6.2). By bounded distortion (4.1) we have $\||h'|^\sigma\|_{(\kappa)} \ll |\sigma|^\kappa \|h'\|^{\sigma-\kappa}_\infty \|h''\|^\kappa_\infty \ll |\sigma|^\kappa \|h'\|^\sigma_\infty$. Moreover, by (4.5) we have $\|h'\|^\kappa_\infty \ll \rho^{\kappa r}$. Thus (7.5) follows by (4.3) and (7.4). □

Lemma 7.5. *The map $\mathcal{F}_\Theta$ is topologically mixing.*

Proof. Let r_0 be given by Hypothesis (5.2). Let $\mathcal{V} \subset K$ be a non-empty open arc and $\vartheta_1 \in \Theta$. We wish to show that for some $m \geq 1$ depending on $\mathcal{V}$, and all $n \geq m$, we have $\mathcal{F}^n_\Theta(\mathcal{V} \times \{\vartheta_1\}) = K_\Theta$. Since the diameter of a fundamental arc $[w_0]$, $w_0 = a_1 \cdots a_n \in \Sigma^*$, tends uniformly to 0 as $n \to \infty$, we deduce that $\mathcal{V}$ contains such an interval with $n \geq 1$.

Let $x \in K$ and $\vartheta_2 \in \Theta$. We apply the hypothesis (5.2) with $b_1 = a_n$, $b_2 \in \Sigma$ the letter such that $x \in [b_2]$, and

$$\vartheta_0 = \vartheta_1^{-1}\vartheta_2\vartheta_{a_1\cdots a_{n-1}}(a_n)^{-1}.$$

We deduce the existence of w with $a_n w b_2 \in \Sigma^*$ such that

$$\vartheta_{a_n w}(b_2) = \vartheta_1^{-1}\vartheta_2\vartheta_{a_1\cdots a_{n-1}}(a_n)^{-1} \quad \text{and} \quad \ell_K(a_n w b_2) = r_0.$$

Let $h = g_{w_0 w}$. By the definition (3.1) of ℓ_K we have

$$m := \ell_K(w_0 w b_2) = \ell_K(w_0) + r_0 - 1,$$

which depends neither on x nor on ϑ_2. Then, we have

$$\mathcal{F}^m_\Theta(h(x), \vartheta_1) = (x, \vartheta_1\vartheta_{w_0 w}(b_2)).$$

However by the cocycle property we have

$$\vartheta_1\vartheta_{w_0 w}(b_2) = \vartheta_1\vartheta_{a_n w}(b_2)\vartheta_{a_1\cdots a_{n-1}}(a_n) = \vartheta_2$$

by construction, and so

$$\mathcal{F}^m_\Theta(h(x), \vartheta_1) = (x, \vartheta_2).$$

Since we have $h(x) \subset [w_0] \subset \mathcal{V}$, (x, ϑ_2) is arbitrary and m is independent of (x, ϑ_2), we deduce

$$\mathcal{F}_\Theta^m(\mathcal{V} \times \{\vartheta_1\}) = K_\Theta$$

as required. □

The proof of Proposition 7.3 follows now by standard arguments, see [Bal00, Theorem 1.5]. Lemmas 7.2 and 7.4 (Lasota-Yorke inequalities) show that Hennion's criterion [Hen93] is statisfied. This implies quasi-compactness of $\mathscr{L}_{\sigma,0}$ acting on H^κ. The positivity of some eigenvalue $\lambda_{\sigma,0}$ of largest modulus and its eigenfunction $\xi_{\sigma,0}$ follows from the fact that $\mathscr{L}_{\sigma,0}$ is weighted by the positive real-valued function $g := |h'|^\sigma$.

For the second item, we use Lemma 7.5 (topological mixing) to conclude that $\lambda_{\sigma,0}$ is a unique and simple eigenvalue of largest modulus by density and convergence arguments. In particular, by change of variable, the operator $\mathscr{L}$ admits 1 as dominant eigenvalue with the associated eigenfunction ξ. This spectral gap yields a spectral decomposition

$$\mathscr{L}_{\sigma,0} = \lambda_{\sigma,0}\mathscr{P}_{\sigma,0} + \mathscr{N}_{\sigma,0}$$

where the eigenprojection $\mathscr{P}_{\sigma,0}$ has an image spanned by $\xi_{\sigma,0}$ and $\operatorname{spr}(\mathscr{N}_{\sigma,0}) < \lambda_{\sigma,0}$ by the quasi-compactness. The spectral projection $\mathscr{P}_{\sigma,0}$ has an integral form similar to (7.3).

7.3. **Perturbation estimates.**

7.3.1. *Estimates relative to $\mathcal{F}_K$.* We recall the definition (5.9) and its equivalent counterpart (5.11). For all $m \geq 0$, the operator on $L^\infty(K_\Theta)$

$$f \mapsto \mathscr{L}[(\log|\mathcal{F}_K'|)^m f](y, \vartheta_1) = \sum_{\substack{h \in \mathcal{H} \\ y \in \operatorname{Dom}(h)}} (-\log|h'(y)|)^m |h'(y)| f(h(y), \vartheta_1\vartheta_h(y))$$

is well-defined by Lemma 6.2. In the same way, the operators

$$f \mapsto \mathscr{L}[\Phi_K(\log|\mathcal{F}_K'|)^m f](y, \vartheta_1) = \sum_{\substack{h \in \mathcal{H} \\ y \in \operatorname{Dom}(h)}} \vartheta_1 \boldsymbol{\phi}_h(y)(-\log|h'(y)|)^m |h'(y)| f(h(y), \vartheta_1\vartheta_h(y)),$$

$$f \mapsto \mathscr{L}[\Phi_K \cdot \Phi_K^T f](y, \vartheta_1) = \sum_{\substack{h \in \mathcal{H} \\ y \in \operatorname{Dom}(h)}} \|\vartheta_1 \boldsymbol{\phi}_h(y)\|_2^2 |h'(y)| f(h(y), \vartheta_1\vartheta_h(y))$$

are well-defined provided that Hypothesis (M_K) holds with $\alpha > 1$ and $\alpha > 2$ respectively. Here $\|\cdot\|_2$ is the Euclidean norm.

Finally, we state some perturbation estimates and expansions.

Lemma 7.6. *Assume Hypothesis (M_K). For all s and $\boldsymbol{t}$ with $|s-1|$ and $|\boldsymbol{t}|$ sufficiently small, we have*

$$\|\mathscr{L}_{s,\boldsymbol{t}} - \mathscr{L}\|_\infty \ll |s-1| + |\boldsymbol{t}| + |\boldsymbol{t}|^{\alpha-\varepsilon}, \tag{7.6}$$

$$\|\mathscr{L}_{s,\boldsymbol{t}} - \mathscr{L}\|_\kappa \ll |s-1| + |\boldsymbol{t}| + |\boldsymbol{t}|^{\alpha-\varepsilon}. \tag{7.7}$$

Proof. Let $\boldsymbol{t}$ and $s-1$ be sufficiently small. We first prove (7.6). We have $e^y = 1 + O(|y|^{\alpha-\varepsilon} + |y|)$ for $y \in i\mathbb{R}$, so that from the definition (5.13), Proposition 5.4 and Lemma 6.2 we obtain (7.6) by Hölder's inequality.

We now move to (7.7). Let $f \in H^\kappa$. By (7.6) it suffices to bound $\|\mathscr{L}_{s,\boldsymbol{t}}[f] - \mathscr{L}[f]\|_{(\kappa)}$. By (6.1)-(6.2) and splitting the differences, we have

$$\begin{aligned}
&\|(|h'(y)|^s e^{i\langle \boldsymbol{t}, \vartheta_1 \boldsymbol{\phi}_h\rangle} - |h'(y)|) f(h(y), \vartheta_1\vartheta_h(y))\|_{(\kappa)} \\
&\qquad \ll \|(|h'(y)|^s e^{i\langle \boldsymbol{t}, \vartheta_1 \boldsymbol{\phi}_h\rangle} - |h'(y)|)\|_\infty \|f\|_{(\kappa)} + \||h'(y)|^s\|_{(\kappa)} \|(\mathrm{e}^{i\langle \boldsymbol{t}, \vartheta_1 \boldsymbol{\phi}_h\rangle} - 1)\|_\infty \|f\|_\infty \\
&\qquad\quad + \||h'(y)|^s - |h'(y)|\|_{(\kappa)} \|f\|_\infty + \||h'(y)|^\sigma\|_\infty \|(e^{i\langle \boldsymbol{t}, \vartheta_1 \boldsymbol{\phi}_h\rangle} - 1)\|_{(\kappa)} \|f\|_\infty.
\end{aligned}$$

The first three summands on the right can be dealt as for (7.6), where for the third one we use that $\||h'|^s - |h'|\|_{(\kappa)} \ll |s-1|(\|h'\|_\infty + \|h'\|_\infty^\sigma)|\log|h'||$ which follows, as in the proof of Lemma 7.4, from $|h'|^s - |h'| = \int_1^s |h'|^w \log|h'|\,dw$. It remains to deal with the last summand. We let

$$\kappa_1 = \kappa/\min(1, \alpha-\varepsilon).$$

Notice that clearly $\kappa_1 \geq \kappa$ and, by (7.1), we also have $\kappa_1 \leq \kappa_0/2 \leq 1/2$. By (6.3) and (6.4) we then have

$$\begin{aligned}
\|e^{i\langle \boldsymbol{t}, \vartheta_1 \boldsymbol{\phi}_h\rangle} - 1\|_{(\kappa)} = \|e^{i\langle \boldsymbol{t}, \vartheta_1 \boldsymbol{\phi}_h\rangle}\|_{(\kappa)} &\ll |\boldsymbol{t}|^{\kappa/\kappa_1} \|\boldsymbol{\phi}_h\|_{(\kappa_1),\mathrm{Dom}(h)}^{\kappa/\kappa_1} \\
&\leq |\boldsymbol{t}|^{\kappa/\kappa_1} \|\boldsymbol{\phi}_h\|_\infty^{\kappa/\kappa_1 - \kappa/\kappa_0} \|\boldsymbol{\phi}_h\|_{(\kappa_0),\mathrm{Dom}(h)}^{\kappa/\kappa_0}
\end{aligned}$$

Next, we let

$$(\kappa/\kappa_1 - \kappa/\kappa_0)p = \alpha, \qquad \kappa/\kappa_0 q = \min(\lambda, \alpha/2, 1),$$

so that by (7.1), we have

$$1 - \frac{1}{p} - \frac{1}{q} \geq \frac{\alpha - \min(1, \alpha-\varepsilon)}{2\alpha} > 0,$$

and let $r \in (1,\infty)$ be such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Then, by Hölder inequality we have

$$\begin{aligned}
&\sum_{h\in\mathcal{H}} \||h'(y)|^\sigma\|_\infty \|(e^{i\langle \boldsymbol{t}, \vartheta_1 \boldsymbol{\phi}_h\rangle} - 1)\|_{(\kappa)} \|f\|_\infty \\
&\qquad \leq |\boldsymbol{t}|^{\min(1,\alpha-\varepsilon)} \sum_{h\in\mathcal{H}} \|h'(y)\|_\infty^\sigma \|\boldsymbol{\phi}_h\|_\infty^{\kappa/\kappa_1 - \kappa/\kappa_0} \|\boldsymbol{\phi}_h\|_{(\kappa_0),\mathrm{Dom}(h)}^{\kappa/\kappa_0} \\
&\qquad \leq |\boldsymbol{t}|^{\min(1,\alpha-\varepsilon)} \Big(\sum_{h\in\mathcal{H}} \|h'(y)\|_\infty^{1+r(\sigma-1)}\Big)^{1/r} \Big(\sum_{h\in\mathcal{H}} \|h'(y)\|_\infty \|\boldsymbol{\phi}_h\|_\infty^\alpha\Big)^{1/p} \\
&\qquad\quad \times \Big(\sum_{h\in\mathcal{H}} \|h'(y)\|_\infty \|\boldsymbol{\phi}_h\|_{(\kappa_0),\mathrm{Dom}(h)}^{\min(\lambda,\alpha/2,1)}\Big)^{1/q} \\
&\qquad \ll |\boldsymbol{t}|^{\min(1,\alpha-\varepsilon)},
\end{aligned}$$

by Proposition 5.4 and (6.5)-(6.6), provided that $\sigma - 1$ is small enough in terms of ε and α. □

If $\alpha > 2$ we can refine (7.6) further.

Lemma 7.7. *Assume $\alpha > 2$. There exists $t_0 > 0$ such that for $|s-1| \ll |\boldsymbol{t}| < t_0$, we have*

$$\begin{aligned}
\mathscr{L}_{s,\boldsymbol{t}}[f] = \mathscr{L}[f] &+ i\langle \boldsymbol{t}, \mathscr{L}[\Phi_K f]\rangle - (s-1)\mathscr{L}[\log|\mathcal{F}_K'| f] + \\
&- \tfrac{1}{2}\boldsymbol{t}^T \mathscr{L}[\Phi_K \cdot \Phi_K^T f]\boldsymbol{t} - i(s-1)\langle \boldsymbol{t}, \mathscr{L}[\Phi_K \log|\mathcal{F}_K'| f]\rangle \\
&+ \tfrac{1}{2}(s-1)^2 \mathscr{L}[\log|\mathcal{F}_K'|^2 f] + O(|\boldsymbol{t}|^3 + |\boldsymbol{t}|^{\alpha-\varepsilon})
\end{aligned}$$

in $L^\infty(K_\Theta)$.

Proof. Write $\mathscr{L}_{s,t}[f] = \mathscr{L}[\mathrm{e}^{i\langle \boldsymbol{t}, \Phi_K\rangle}|h'|^{s-1}f]$. First we expand $|h'|^{s-1}$ to third order, which, by Lemma 5.4, gives an error term $O(|s-1|)^3 = O(|\boldsymbol{t}|^3)$ for t_0 small enough. Then we expand the exponential by $e^{it} = 1 + it - t^2/2 + O(|t|^{\min(3,\alpha-\varepsilon)})$ for real t. The ensuing error terms are bounded by

$$\sum_{h\in\mathcal{H}} |h'|^\sigma (1 + |\log |h'||^2)(|\boldsymbol{t}| \|\boldsymbol{\phi}_h\|_{\infty,\mathrm{Dom}(h)})^{\min(3,\alpha-\varepsilon)}.$$

where $\sigma = \mathfrak{Re}(s)$. By the Hölder inequality, and applying Lemmas 5.4 and 6.2, we have that this is $O(|\boldsymbol{t}|^{\min(3,\alpha-\varepsilon)})$ provided that t_0 is small enough. □

7.3.2. *Estimates relative to $\mathcal{F}_c$.* In this subsection, we assume the hypotheses (M_c). Our goal is to provide technical estimates needed in order to rephrase the parameters of the limit law in Theorem 1.3 in terms of $\mathcal{F}_c$ and its inverse branches, whenever possible. We recall the definition (6.8) of Φ_c and its relation (6.7) with Φ_K.

The following lemma relates $\mathscr{L}_{1,\boldsymbol{t}}$ to $\mathscr{C}_{1,\boldsymbol{t}}$, using the decomposition given in Lemma 3.3.

Lemma 7.8. *Assume Hypothesis (M_c). Then, for $f \in L^\infty(K_\Theta)$ we have*

$$\begin{aligned}\int_{K_\Theta} (\mathscr{L}_{1,\boldsymbol{t}} - \mathscr{L})[f]\,\mathrm{d}\nu &= \int_{K_\Theta} \big((\mathscr{C}_{1,\boldsymbol{t}} - \mathscr{C}) + \mathscr{C}(\mathrm{Id} - \mathscr{C}^\sharp)^{-1}(\mathscr{C}^\sharp_{1,\boldsymbol{t}} - \mathscr{C}^\sharp)\big)(\mathrm{Id} - \mathscr{C}^\sharp)^{-1}[f]\,\mathrm{d}\nu \\ &\quad + \int_{K_\Theta} \big((\mathscr{C}_{1,\boldsymbol{t}} - \mathscr{C}) + \mathscr{C}(\mathrm{Id} - \mathscr{C}^\sharp)^{-1}(\mathscr{C}^\sharp_{1,\boldsymbol{t}} - \mathscr{C}^\sharp)\big)(\mathscr{C}^\sharp_{1,\boldsymbol{t}} - \mathscr{C}^\sharp)(\mathrm{Id} - \mathscr{C}^\sharp)^{-1}[f]\,\mathrm{d}\nu \\ &\quad + O\big(|\boldsymbol{t}|^2 + |\boldsymbol{t}|^{2\alpha-2\varepsilon}\big)\end{aligned}$$

Proof. First, we observe that proceeding as in (7.6) one has that (6.9) implies

$$\|\mathscr{C}_{1,\boldsymbol{t}}(\mathscr{C}^\sharp_{1,\boldsymbol{t}} - \mathscr{C}^\sharp)\|_\infty + \|\mathscr{C}(\mathscr{C}^\sharp_{1,\boldsymbol{t}} - \mathscr{C}^\sharp)\|_\infty + \|\mathscr{C}^\sharp(\mathscr{C}^\sharp_{1,\boldsymbol{t}} - \mathscr{C}^\sharp)\|_\infty \ll |\boldsymbol{t}| + |\boldsymbol{t}|^{\alpha-\varepsilon}, \tag{7.8}$$

$$\int_{\partial\mathbb{D}_\Theta} |(\mathscr{C}_{1,\boldsymbol{t}} - \mathscr{C})[f]|\,\mathrm{d}\nu \ll |\boldsymbol{t}| + |\boldsymbol{t}|^{\alpha-\varepsilon} \qquad \forall f \in L^\infty(\partial\mathbb{D}_\Theta). \tag{7.9}$$

Recalling (6.11), a quick computation shows that on bounded functions supported on K_Θ we have

$$\begin{aligned}\mathscr{L}_{1,\boldsymbol{t}} - \mathscr{L} &= \mathscr{C}_{1,\boldsymbol{t}}(\mathrm{Id} - \mathscr{C}^\sharp_{1,\boldsymbol{t}})^{-1} - \mathscr{C}(\mathrm{Id} - \mathscr{C}^\sharp)^{-1} \\ &= \big((\mathscr{C}_{1,\boldsymbol{t}} - \mathscr{C}) + \mathscr{C}(\mathrm{Id} - \mathscr{C}^\sharp)^{-1}(\mathscr{C}^\sharp_{1,\boldsymbol{t}} - \mathscr{C}^\sharp)\big)(\mathrm{Id} - \mathscr{C}^\sharp)^{-1} \\ &\quad + \big((\mathscr{C}_{1,\boldsymbol{t}} - \mathscr{C}) + \mathscr{C}(\mathrm{Id} - \mathscr{C}^\sharp)^{-1}(\mathscr{C}^\sharp_{1,\boldsymbol{t}} - \mathscr{C}^\sharp)\big)(\mathscr{C}^\sharp_{1,\boldsymbol{t}} - \mathscr{C}^\sharp)(\mathrm{Id} - \mathscr{C}^\sharp)^{-1} \\ &\quad + \big((\mathscr{C}_{1,\boldsymbol{t}} - \mathscr{C}) + \mathscr{C}(\mathrm{Id} - \mathscr{C}^\sharp)^{-1}(\mathscr{C}^\sharp_{1,\boldsymbol{t}} - \mathscr{C}^\sharp)\big)\mathscr{C}_{1,\boldsymbol{t}}(\mathrm{Id} - \mathscr{C}^\sharp)^{-1}(\mathscr{C}^\sharp_{1,\boldsymbol{t}} - \mathscr{C}^\sharp)(\mathrm{Id} - \mathscr{C}^\sharp)^{-1}.\end{aligned}$$

We apply this to f, and integrate over K_Θ and the result follows by estimating the contribution of the last line by (7.8)-(7.9). □

7.4. **Dolgopyat-type estimate for the twists.** The goal of this section is to show the existence of a strip $\{s \in \mathbb{C}, \mathfrak{Re}(s) > \sigma\}$ for some $\sigma < 1$, in which $\mathscr{L}_{s,\boldsymbol{t}}$ is contracting in a quantitative sense. To this end we use a method originating from work of Dolgopyat [Dol98]. This method relies on a property of the map $\mathcal{F}_K$ known as uniform non-integrability (UNI). Dolgopyat's paper contains two arguments. One method (Section 5 *ibid*) relies on a stronger hypothesis, and is simpler to implement. The other method (Sections (6–8) *ibid.*) relies on a weaker condition, but is much more involved. The conditions were called respectively strong UNI and weak UNI in [BV05b, BV05a].

The generalization of Dolgopyat's argument to a case with countably many branches was obtained by Baladi–Vallée [BV05a], under the weak UNI condition. The specific case of the Gauss map was included in the earlier work of Baladi-Vallée [BV05b], and there the strong UNI condition was used.

In the following, we show that the strong UNI condition also holds in our case (this implies automatically the weak UNI condition), using the explicit construction of the natural extension from Section 3.4. We then justify that the strong UNI argument due to Dolgopyat [Dol98] and Baladi–Vallée [BV05b] hold in our situation.

7.4.1. *The strong UNI condition.* Our aim in this section is to show the "strong UNI" property of $\mathcal{F}_K$ acting on the circle; see [BV05b, p. 357] for the case of the Gauss map. Given $n \geq 1$ and $h, k \in \mathcal{H}^n$, we denote

$$\Delta(h,k) := \inf_{x \in \mathrm{Dom}(h) \cap \mathrm{Dom}(k)} \left| \frac{h''}{h'}(x) - \frac{k''}{k'}(x) \right|. \tag{7.10}$$

Proposition 7.9. *There exists a constant $\rho < 1$ such that for all $\eta > 0$, $n \geq 1$ and $k \in \mathcal{H}^n$, we have*

$$\sum_{\substack{h \in \mathcal{H}^n \\ \Delta(h,k) \leq \eta}} \|h'\|_\infty \ll \eta + \rho^n.$$

The arguments we propose is an extension of those in [BV05b, Section 3.4]. The question is translated entirely in terms of a dual dynamical system, which is also Markov and hyperbolic, whose inverse branches are $\{h^{-1}, h \in \mathcal{H}^*\}$: then, by means of (4.2), this allows to translate the condition $\Delta(h,k) \leq \eta$ into a condition of type $h(x) \in I$ for some interval I of length $O(\eta)$. This is then implemented as an input test function for the dual transfer operator. That such a system exists is related to the existence of the natural extension going backwards in the coding, see the comment in [BV05b, p. 363]. This is the main reason we needed to explicit the natural extensions for $\mathcal{F}_K$ in Section 3.4.

Proof. We let $\rho \in (\mathrm{spr}(\mathscr{N}), 1)$, which is allowed by Proposition 7.3, and suppose ρ is close enough to 1 that it satisfies also Lemma 4.3.

For all $a \in \Sigma$, we let x_a be an arbitrary point of $K \cap (\partial\mathbb{D} \smallsetminus [\widehat{a}])$. By (4.3), for all $h \in \mathcal{H}^*$ we have

$$\|h'\|_\infty \asymp \sum_{\substack{(b_1,b_2) \text{ separated} \\ [b_1,b_2] \subset \mathrm{Dom}(h)}} \|h'\|_{\infty,[b_1,b_2]} \asymp \sum_{\substack{(b_1,b_2) \text{ separated} \\ [b_1,b_2] \subset \mathrm{Dom}(h)}} |h'(g_{b_1} g_{b_2} x_{b_2})|.$$

Next we note that

$$|h'(g_{b_1 b_2} x_{b_2})| = \frac{|(h \circ g_{b_1 b_2})'(x_{b_2})|}{|g'_{b_1 b_2}(x_{b_2})|} \asymp |(h \circ g_{b_1 b_2})'(x_{b_2})|.$$

For all $(a_2, b_1) \in \Sigma$, we define

$$W := \{(a_1, a_2, w, b_1, b_2) \in \Sigma^2 \times \Sigma^* \times \Sigma^2 \mid (a_1,a_2) \text{ and } (b_1,b_2) \text{ separated}, \ell_K(a_2 w b_1) = n\}$$
$$W' := \{(h, b_1, b_2) \in \mathcal{H}^n \times \Sigma^2 \mid \mathrm{Im}(h) \subset K, (b_1,b_2) \text{ separated}, [b_1,b_2] \subset \mathrm{Dom}(h)\}.$$

Then the map

$$W \ni (a_1, a_2, w, b_1, b_2) \mapsto (h_{a_1 a_2 w, n}, b_1, b_2) \in W' \tag{7.11}$$

is a bijection and so, from the above discussion, we have

$$\sum_{\substack{h\in\mathcal{H}^n\\ \Delta(h,k)\leq\eta}} \|h'\|_\infty \asymp \sum_{\substack{(a_1,a_2)\\ (b_1,b_2)\\ \text{separated}}} \sum_{\substack{w\\ \ell_K(a_2wb_1)=n\\ \Delta(g_{a_1a_2w},k)\leq\eta}} \big|g'_{a_1a_2wb_1b_2}(x_{b_2})\big|.$$

We first transform the condition involving $\Delta(h,k)$. By (4.2), we have

$$\Delta(h,k) \gg \big|g^{-1}_{a_1a_2w}(0) - k^{-1}(0)\big|.$$

It follows that there exists a set $I\subset\mathbb{C}$ of diameter $O(\eta)$, depending only on k, such that

$$\Delta(h,k)\leq\eta \implies g^{-1}_{a_1a_2w}(0)\in I.$$

Define

$$J_1 := \bigcup_{\substack{(b_1,b_2)\\ \text{separated}}} g^{-1}_{b_1b_2}(I),$$

so that J_1 a finite union of sets of diameter $O(\eta)$. For all b_1, b_2 separated we then have

$$\Delta(h,k)\leq\eta \implies g^{-1}_{a_1a_2wb_1b_2}(0)\in J_1.$$

Finally, using (4.3) and the fact that $|g'(0)| = |(g^{-1})'(0)|$ for $g\in\mathrm{SU}(1,1)$, we have

$$\big|g'_{a_1a_2wb_1b_2}(x_{b_2})\big| \asymp \big|g'_{a_1a_2wb_1b_2}(0)\big| = \big|(g^{-1}_{a_1a_2wb_1b_2})'(0)\big|.$$

From the above, we conclude that

$$\sum_{\substack{h\in\mathcal{H}^n\\ \Delta(h,k)\leq\eta}} \|h'\|_\infty \ll \sum_{\substack{(a_1,a_2)\\ (b_1,b_2)\\ \text{separated}}} \sum_{\substack{w\\ \ell_K(a_2wb_1)=n\\ g^{-1}_{a_1a_2wb_1b_2}(0)\in J_1}} \big|(g^{-1}_{a_1a_2wb_1b_2})'(0)\big|.$$

We may now use the fact that the set W is invariant under the involution

$$(a_1,a_2,w,b_1,b_2)\mapsto(\widehat{b}_2,\widehat{b}_1,\widehat{w},\widehat{a}_2,\widehat{a}_1),$$

to the effect that

$$\sum_{\substack{(a_1,a_2)\\ (b_1,b_2)\\ \text{separated}}} \sum_{\substack{w\\ \ell_K(a_2wb_1)=n\\ g^{-1}_{a_1a_2wb_1b_2}(0)\in J_1}} \big|(g^{-1}_{a_1a_2wb_1b_2})'(0)\big| = \sum_{\substack{(a_1,a_2)\\ (b_1,b_2)\\ \text{separated}}} \sum_{\substack{w\\ \ell_K(a_2wb_1)=n\\ g_{a_1a_2wb_1b_2}(0)\in J_1}} \big|g'_{a_1a_2wb_1b_2}(0)\big|.$$

We now use again (4.3) and (4.4), obtaining

$$\sum_{\substack{(a_1,a_2)\\ (b_1,b_2)\\ \text{separated}}} \sum_{\substack{w\\ \ell_K(a_2wb_1)=n\\ g_{a_1a_2wb_1b_2}(0)\in J_1}} \big|g'_{a_1a_2wb_1b_2}(0)\big| \ll \sum_{\substack{(a_1,a_2)\\ (b_1,b_2)\\ \text{separated}}} \sum_{\substack{w\\ \ell_K(a_2wb_1)=n\\ g_{a_1a_2wb_1b_2}(x_{b_2})\in J_2}} \big|g'_{a_1a_2wb_1b_2}(x_{b_2})\big|$$

for some set J_2 which, by (4.5), is a finite union of sets of diameter $O(\eta+\rho^n)$. In the last expression, we have

$$\big|g'_{a_1a_2wb_1b_2}(x_{b_2})\big| = \big|g'_{a_1a_2w}(g_{b_1b_2}x_{b_2})g'_{b_1b_2}(x_{b_2})\big| \ll \big|g'_{a_1a_2w}(g_{b_1b_2}x_{b_2})\big|,$$

and therefore, letting $y_{b_1b_2} := g_{b_1b_2}x_{b_2}$ and using the bijection (7.11),

$$\begin{aligned}\sum_{\substack{h\in\mathcal{H}^n\\ \Delta(h,k)\le\eta}} \|h'\|_\infty &\ll \sum_{\substack{(a_1,a_2)\\(b_1,b_2)\\ \text{separated}}} \sum_{\substack{w\\ \ell_K(a_2wb_1)=n\\ g_{a_1a_2w}(y_{b_1b_2})\in J_2}} \left|g'_{a_1a_2w}(y_{b_1b_2})\right| \\ &= \sum_{\substack{(b_1,b_2)\\ \text{separated}}} \sum_{\substack{h\in\mathcal{H}^n\\ y_{b_1b_2}\in\mathrm{Dom}(h)}} |h'(y_{b_1b_2})|\cdot \mathbf{1}_{J_2}(h(y_{b_1b_2})) \\ &= \sum_{\substack{(b_1,b_2)\\ \text{separated}}} \mathscr{L}^n[\mathbf{1}^*_{J_2}](y_{b_1b_2}), \end{aligned} \tag{7.12}$$

where $\mathbf{1}^*_{J_2}(x,\vartheta_0) := \mathbf{1}_{J_2}(x)$ for all $(x,\vartheta_0)\in K_\Theta$. To estimate $\mathscr{L}^n[\mathbf{1}^*_{J_2}]$, we first upper-bound the map $\mathbf{1}_{J_2}$ by a smooth, bounded map ψ on $\partial\mathbb{D}$ such that

$$\int_K \psi = O(\eta+\rho^{n/2}), \qquad \|\psi'\|_\infty = O(\rho^{-n/2}).$$

Letting $\psi^*(x,\vartheta_0) := \psi(x)$, we deduce $\|\psi^*\|_\kappa = \|\psi\|_\infty + \|\psi\|_{(\kappa)} \ll 1+\|\psi'\|_\infty^\kappa \ll \rho^{-\kappa n/2}$. By Proposition 7.3.(iii), it follows that

$$\mathscr{L}^n[\psi^*](y) = \Big(\int_K \psi\Big)\xi(y) + O(\rho^n\|\psi^*\|_\kappa) \ll \eta+\rho^{n/2}$$

uniformly in $y\in K$. Since $\mathbf{1}_{J_2}\le\psi$ and the sum in (7.12) over (b_1,b_2) is finite, we deduce the claimed bound, up to the value of ρ. □

7.4.2. *Estimates on the quasi-inverse for large twists.* We start with an uniform version of (6.6) for inverse branches of $\mathcal{H}^n$.

Lemma 7.10. *Assume* (6.6) *holds. Then uniformly in* $n\ge1$ *we have*

$$\sum_{h\in\mathcal{H}^n} \|h'\|_\infty\cdot\|\boldsymbol{\phi}_h\|^\lambda_{(\kappa_0),\mathrm{Dom}(h)} \ll 1. \tag{7.13}$$

Proof. Writing each $h\in\mathcal{H}^n$ as $h = h_1\circ\cdots\circ h_n\in\mathcal{H}^n$ with $h_i\in\mathcal{H}$, we have by (5.12)

$$\|\boldsymbol{\phi}_h\|_{(\kappa_0),\mathrm{Dom}(h)} \le \sum_{i=1}^n \|\boldsymbol{\phi}_{h_i}(h_{i+1}\circ\cdots\circ h_n(y))\|_{(\kappa_0),\mathrm{Dom}(h)} \ll \sum_{i=1}^n \|\boldsymbol{\phi}_{h_i}\|_{(\kappa_0),\mathrm{Dom}(h_i)}\rho^{n-i}$$

by (6.2) and (4.5). If $\lambda\ge1$ this implies by Hölder's inequality

$$\|\boldsymbol{\phi}_h\|^\lambda_{(\kappa_0),\mathrm{Dom}(h)} \le \Big(\sum_{i=1}^n \|\boldsymbol{\phi}_{h_i}\|^\lambda_{(\kappa_0),\mathrm{Dom}(h_i)}\rho^{(n-i)}\Big)\Big(\sum_{i=1}^n \rho^{(n-i)}\Big)^{\lambda-1} \ll \sum_{i=1}^n \|\boldsymbol{\phi}_{h_i}\|^\lambda_{(\kappa_0),\mathrm{Dom}(h_i)}\rho^{(n-i)}$$

and the same bounds hold trivially for $\lambda<1$, save for the fact that ρ is replaced by ρ^λ. Thus, enlarging the value of $\rho\in(0,1)$ if needed, we have

$$\begin{aligned}\sum_{h\in\mathcal{H}^n} \|h'\|_\infty\cdot\|\boldsymbol{\phi}_h\|^\lambda_{(\kappa_0),\mathrm{Dom}(h)} &\ll \sum_{\substack{n_1+n_2=n-1,\\ n_1,n_2\ge0}} \rho^{n_2} \sum_{\substack{h_1\in\mathcal{H}^{n_1},h\in\mathcal{H},h_2\in\mathcal{H}^{n_2},\\ \mathrm{Im}(h)\subset\mathrm{Dom}(h_1),\\ \mathrm{Im}(h_2)\subset\mathrm{Dom}(h)}} \|(h_1\circ h\circ h_2)'\|_\infty\|\boldsymbol{\phi}_h\|^\lambda_{(\kappa_0),\mathrm{Dom}(h)} \\ &\ll \sum_{\substack{n_1+n_2=n-1,\\ n_1,n_2\ge0}} \rho^{n_2} \ll 1,\end{aligned}$$

by (6.6) and (7.4). □

Next, we prove a Lasota–Yorke estimate for $\mathscr{L}_{s,\boldsymbol{t}}$.

Lemma 7.11. *Let ρ be as in Lemma 4.3. Then, there exists $\delta_0 > 0$ such that for all $r \geq 1$, $s = \sigma + i\tau$ with $\sigma \geq 1 - \delta_0$, $\boldsymbol{t} \in \mathbb{R}^d$, and $f \in H^\kappa$, we have*

$$\|\mathscr{L}_{s,\boldsymbol{t}}^r[f]\|_{(\kappa)} \ll A_{2\sigma-1}^{r/2}\big((|\boldsymbol{t}|^{\min(1,\lambda)/2} + |s|^\kappa)\|f\|_\infty + \rho^{\kappa r}\|f\|_{(\kappa)}\big).$$

Proof. Let $\delta_0 := (1-\sigma_0)/2$. As in the proof of (7.5), we have $\||h'|^s\|_{(\kappa)} \ll |s|^\kappa \|h'\|_\infty^\sigma$. Also, since $\kappa/\kappa_0 \leq \min(1,\lambda)/2$, by (6.3) we have

$$\|e^{i\langle \boldsymbol{t}, \vartheta_1 \boldsymbol{\phi}_h\rangle}\|_{(\kappa)} \ll |\boldsymbol{t}|^{\kappa/\kappa_0}\|\boldsymbol{\phi}_h\|_{(\kappa_0),\mathrm{Dom}(h)}^{\kappa/\kappa_0} \ll |\boldsymbol{t}|^{\min(1,\lambda)/2}\big(1 + \|\boldsymbol{\phi}_h\|_{(\kappa_0),\mathrm{Dom}(h)}^{\lambda/2}\big)$$

Thus, by (6.1)-(6.2), and (4.5) we have

$$\|\mathscr{L}_{s,\boldsymbol{t}}^r f\|_{(\kappa)} \ll \sum_{h \in \mathcal{H}^r} \|h'\|_\infty^\sigma \Big(\big(|s|^\kappa + |\boldsymbol{t}|^{\kappa/\kappa_0}\|\boldsymbol{\phi}_h\|_{(\kappa_0),\mathrm{Dom}(h)}^{\kappa/\kappa_0}\big)\|f\|_\infty + \rho^{\kappa r}\|f\|_{(\kappa)}\Big).$$

The claimed bound then follows since by the Cauchy-Schwarz inequality and (7.4) we have

$$\begin{aligned}\|\mathscr{L}_{s,\boldsymbol{t}}^r f\|_{(\kappa)}^2 &\ll A_{2\sigma-1}^r \sum_{h\in\mathcal{H}^r} \|h'\|_\infty \Big(\big(|s|^{2\kappa} + |\boldsymbol{t}|^{2\kappa/\kappa_0}\|\boldsymbol{\phi}_h\|_{(\kappa_0),\mathrm{Dom}(h)}^{2\kappa/\kappa_0}\big)\|f\|_\infty^2 + \rho^{2\kappa r}\|f\|_{(\kappa)}^2\Big) \\ &\ll A_{2\sigma-1}^r\big((|s|^{2\kappa} + |\boldsymbol{t}|^{\min(1,\lambda)})\|f\|_\infty^2 + \rho^{2\kappa r}\|f\|_{(\kappa)}^2\big)\end{aligned}$$

where in the last step we used Lemma 7.10 and (7.2). ☐

Using Lemma 7.11 and Proposition 7.9, we can establish the following L^2-type estimate.

Proposition 7.12. *Let ρ be as in Lemma 4.3. There exists $\delta_0 > 0$ such that for all $\boldsymbol{t} \in \mathbb{R}^d$ and $s = \sigma + i\tau$ satisfying $1 - \delta_0 \leq \sigma \ll 1$ and $|\tau| \geq \rho^{-n}$, we have*

$$\int_{K_\Theta} |\mathscr{L}_{s,\boldsymbol{t}}^n[f]|^2 \,\mathrm{d}\nu \ll A_{2\sigma-1}^n \rho^{\kappa n/2}\|f\|_\infty\big((1 + |\boldsymbol{t}|^{\min(1,\lambda)})\|f\|_\infty + |\tau|^{-\kappa}\|f\|_{(\kappa)}\big) \tag{7.14}$$

for all $f \in H^\kappa$.

Proof. Expanding the square, we have

$$\int_{K_\Theta} |\mathscr{L}_{s,\boldsymbol{t}}^n[f]|^2 d\nu = \sum_{h,k\in\mathcal{H}^n} I(h,k) \tag{7.15}$$

where

$$\begin{aligned} I(h,k) &= \int_{\mathrm{Dom}(h)\cap\mathrm{Dom}(k)\times\Theta} e^{i\tau \log\frac{|h'(y)|}{|k'(y)|}} p_h(y,\vartheta_1)\overline{p_k(y,\vartheta_1)} d\nu, \\ p_h(y,\vartheta_1) &= e^{i\langle \boldsymbol{t}, \vartheta_1\boldsymbol{\phi}_h(y)\rangle}|h'(y)|^\sigma \cdot f(h(y), \vartheta_1\vartheta_h(y)).\end{aligned}$$

We decompose (7.15) into two parts according to the distance (7.10). Let I^- denote the sum over the branches satisfying $\Delta(h,k) \leq \eta$ and I^+ denote the sum over the branches satisfying $\Delta(h,k) > \eta > 0$.

Regarding I^-, we observe that

$$|I(h,k)| \leq \|p_h p_k\|_\infty \ll \|f\|_\infty^2 \|h'\|_\infty^\sigma \|k'\|_\infty^\sigma$$

hence by the Cauchy–Schwarz inequality for δ_0 sufficiently small we have

$$I^- \leq \|f\|_\infty^2 \Big(\sum_{\substack{h,k\in\mathcal{H}^n,\\ \Delta(h,k)\leq\eta}} \|h'\|_\infty \|k'\|_\infty \Big)^{1/2} \Big(\sum_{h,k\in\mathcal{H}^n} \|h'\|_\infty^{2\sigma-1} \|k'\|_\infty^{2\sigma-1} \Big)^{1/2}$$

$$\ll \|f\|_\infty^2 A_{2\sigma-1}^n (\eta+\rho^n)^{1/2} \tag{7.16}$$

by Lemma 7.4 and Proposition 7.9.

For I^+, recalling that ν is the Lebesgue measure on its first component, using the version of Van der Corput Lemma in [BD22, Lemma 4.1] and (6.1) we have

$$I^+ \leq \sum_{\substack{h,k\in\mathcal{H}^n\\ \Delta(h,k)>\eta}} |I(h,k)| \ll \sum_{h,k\in\mathcal{H}^n} \left(\frac{\|p_h p_k\|_\infty}{|\tau|\eta} + \frac{\|p_h\|_{(\kappa)}\|p_k\|_\infty + \|p_h\|_\infty \|p_k\|_{(\kappa)}}{(|\tau|\eta)^\kappa} \right)$$

$$\ll A_{2\sigma-1}^n \Big(\frac{\|f\|_\infty^2}{|\tau|\eta} + \|f\|_\infty \frac{(|\boldsymbol{t}|^{\min(1,\lambda)} + |\sigma|^\kappa)\|f\|_\infty + \rho^{\kappa n}\|f\|_{(\kappa)}}{(|\tau|\eta)^\kappa} \Big) \tag{7.17}$$

where the last line follows as in the proof of Lemma 7.11.

Finally, we pick $\eta=\rho^{n/2}$ so that for $|\tau|\geq\rho^{-n}$ the bounds (7.16)-(7.17) give

$$I^- \ll A_{2\sigma-1}^n \|f\|_\infty^2 \rho^{n/4}$$

$$I^+ \ll A_{2\sigma-1}^n \rho^{\kappa n/2}(1+|\boldsymbol{t}|^{\min(1,\lambda)} + |\sigma|^\kappa)\|f\|_\infty^2 + \frac{\rho^{\kappa n/2}\|f\|_{(\kappa)}\|f\|_\infty}{|\tau|^\kappa}.$$

Inserting these results in (7.15) and recalling $\kappa\leq 1/2$ we get the claimed bound. □

Finally we transfer the L^2-bound in Proposition 7.12 to the infinity norm bound.

Proposition 7.13. *There exist $\delta_0, c, C, \tau_1 > 0$ and $\rho\in(0,1)$ such that for all $\boldsymbol{t}\in\mathbb{R}^d$ and $s=\sigma+i\tau$ satisfying $1-\delta_0\leq\sigma\ll 1$, $|\tau|\geq\tau_1$, and $|\boldsymbol{t}|<|\tau|^c$, we have*

$$\|\mathscr{L}_{s,\boldsymbol{t}}^n[f]\|_\infty \ll \rho^n(\|f\|_\infty + |\tau|^{-\kappa}\|f\|_{(\kappa)})$$

for $n=C\log|\tau|+O(1)$ and all $f\in H^\kappa$.

Proof. We fix ρ_1 such that $\max(\rho^{\kappa/4}, \mathrm{spr}(\mathcal{N})) < \rho_1 < 1$ with ρ as in Lemma 4.3. Also, let $c_0 > \frac{\log\rho_1}{\log\rho}$.

We let $n=\ell_1+\ell_2$. By using the Cauchy–Schwarz inequality and (7.4) we have

$$\|\mathscr{L}_{s,\boldsymbol{t}}^n[f]\|_\infty^2 = \|\mathscr{L}_{s,\boldsymbol{t}}^{\ell_1}[\mathscr{L}_{s,\boldsymbol{t}}^{\ell_2}[f]]\|_\infty^2 \ll A_{2\sigma-1}^{\ell_1}\|\mathscr{L}_{1,0}^{\ell_1}[|\mathscr{L}_{s,\boldsymbol{t}}^{\ell_2}[f]|^2]\|_\infty.$$

By Proposition 7.3.(iii) and (7.3) we have $\mathscr{L}_{1,0}^{\ell_1}[f] \ll \int f\,d\nu + \rho_1^{\ell_1}\|f\|_\kappa$ for $f\in H^\kappa$ and thus we deduce

$$\|\mathscr{L}_{s,\boldsymbol{t}}^n[f]\|_\infty^2 \ll A_{2\sigma-1}^{\ell_1}\left(\int_{K_\Theta} |\mathscr{L}_{s,\boldsymbol{t}}^{\ell_2}[f]|^2 d\nu + \rho_1^{\ell_1}\||\mathscr{L}_{s,\boldsymbol{t}}^{\ell_2}[f]|^2\|_{(\kappa)} \right). \tag{7.18}$$

By Lemma 7.11 and (7.4), and using $A_\sigma \ll A_{2\sigma-1}^{1/2}$ by Cauchy-Schwarz, we have

$$\begin{aligned}\||\mathscr{L}_{s,\boldsymbol{t}}^{\ell_2}[f]|^2\|_{(\kappa)} &\leq \|\mathscr{L}_{s,\boldsymbol{t}}^{\ell_2}[f]\|_{(\kappa)}\|\overline{\mathscr{L}_{s,\boldsymbol{t}}^{\ell_2}[f]}\|_\infty + \|\mathscr{L}_{s,\boldsymbol{t}}^{\ell_2}[f]\|_\infty\|\overline{\mathscr{L}_{s,\boldsymbol{t}}^{\ell_2}[f]}\|_{(\kappa)} \\ &\ll A_{2\sigma-1}^{\ell_2}\|f\|_\infty\big((1+|\boldsymbol{t}|^{\min(1,\lambda)}+|\tau|^\kappa)\|f\|_\infty + \rho_1^{\ell_2}\|f\|_{(\kappa)}\big).\end{aligned}$$

Inserting this bound and (7.14) into (7.18) we deduce that for $|\tau|>\rho_1^{c_0\ell_2}$

$$\begin{aligned}\|\mathscr{L}_{s,\boldsymbol{t}}^n[f]\|_\infty^2 &\ll A_{2\sigma-1}^n\rho_1^{\kappa\ell_2/2}\|f\|_\infty((1+|\boldsymbol{t}|^{\min(1,\lambda)})\|f\|_\infty + |\tau|^{-\kappa}\|f\|_{(\kappa)}) \\ &\quad + A_{2\sigma-1}^n\rho_1^{\ell_1}\|f\|_\infty\big((1+|\boldsymbol{t}|^{\min(1,\lambda)}+|\tau|^\kappa)\|f\|_\infty + \rho_1^{\ell_2}\|f\|_{(\kappa)}\big) \\ &\ll A_{2\sigma-1}^n(\rho_1^{\kappa\ell_2/2}+\rho_1^{\ell_1/2})\|f\|_\infty((1+|\boldsymbol{t}|^{\min(1,\lambda)})\|f\|_\infty + |\tau|^{-\kappa}\|f\|_{(\kappa)}),\end{aligned}$$

where the last bound holds with the additional assumption $\rho_1^{-\ell_1/(2\kappa)} > |\tau|$. Finally, we pick $\ell_1 = c_1 \log|\tau| + O(1)$, $\ell_2 = c_2 \log|\tau| + O(1)$ for some fixed $c_1, c_2 > 0$ chosen so that $\rho_1^{-\ell_1/(2\kappa)} > |\tau| > \rho_1^{c_0\ell_2}$ is satisfied. Notice $n = (1 + c_1/c_2)\ell_2 + O(1) = (1 + c_2/c_1)\ell_1 + O(1)$ and thus $\rho_1^{\kappa\ell_2/2} + \rho_1^{\ell_1/2} \ll \rho_2^n$ with $\rho_2 = \max(\rho_1^{\kappa/(2+2c_1/c_2)}, \rho_1^{1/(2+2c_2/c_1)})$. By continuity of A_σ we can pick $\delta > 0$ small enough so that $A_{2\sigma-1} < \rho_2^{-1/2}$. With these assumptions we then have

$$\|\mathscr{L}_{s,\boldsymbol{t}}^n[f]\|_\infty^2 \ll \rho_2^{n/2}\|f\|_\infty((1 + |\boldsymbol{t}|^{\kappa/\kappa_0})\|f\|_\infty + |\tau|^{-\kappa}\|f\|_{(\kappa)}).$$

Finally, we notice that $\rho_2^{n/4} \ll |\tau|^{-c_3}$ for some $c_3 > 0$. Thus, assuming $|\boldsymbol{t}| \ll |\tau|^c$ for some small enough $c > 0$ we deduce the claimed bound. □

Proposition 7.14. *There exists $c, \delta_0, \tau_1 > 0$ and $\rho \in (0, 1)$ such that for $s = \sigma + i\tau$ and $\boldsymbol{t} \in \mathbb{R}^d$ satisfying*

$$\sigma \geq 1 - \delta_0, \qquad |\tau| \geq \tau_1, \qquad |\boldsymbol{t}| \leq |\tau|^c$$

the operator $\mathscr{L}_{s,\boldsymbol{t}}$ acting on H^κ has spectral radius at most ρ. More precisely, there exists $C > 0$ such that for

$$\|\cdot\|_{1,\tau} := \|\cdot\|_\infty + (1 + |\tau|^{-\kappa})\|\cdot\|_{(\kappa)} \tag{7.19}$$

and under the above conditions

$$\sum_{n\geq 0} \|\mathscr{L}_{s,\boldsymbol{t}}^n[f]\|_{1,\tau} \ll |\tau|^C\|f\|_{1,\tau} \qquad \forall f \in H^\kappa. \tag{7.20}$$

Proof. Let $\delta_0, c, C, \tau_1, n$, $\boldsymbol{t}$ and s as in Proposition 7.13. For $f \in H_\kappa$ and some $\rho \in (0, 1)$ we have

$$\begin{aligned}\|\mathscr{L}_{s,\boldsymbol{t}}^{2n}[f]\|_{1,\tau} &\ll A_{2\sigma-1}^{n/2}\big(1 + |\boldsymbol{t}|^{\min(1,\lambda)/2}/|\tau|^\kappa\big)\|\mathscr{L}_{s,\boldsymbol{t}}^n[f]\|_\infty + \rho^n|\tau|^{-\kappa}\|\mathscr{L}_{s,\boldsymbol{t}}^n[f]\|_{(\kappa)} \\ &\ll \rho^n A_{2\sigma-1}^{n/2}\big(1 + |\boldsymbol{t}|^{\min(1,\lambda)/2}/|\tau|^\kappa\big)\|f\|_{1,\tau}\end{aligned}$$

where in both lines we applied Proposition 7.13 and Lemma 7.11. Reducing δ_0 and c_1 and enlarging τ_1, if needed, we then deduce

$$\|\mathscr{L}_{s,\boldsymbol{t}}^{2n}[f]\|_{1,\tau} \leq \rho^{n/2}\|f\|_{1,\tau}$$

This then implies that $\mathscr{L}_{s,\boldsymbol{t}}$ acting on $(H^\kappa, \|\cdot\|_{1,\tau})$ satisfies $\mathrm{spr}(\mathscr{L}_{s,\boldsymbol{t}}) \leq \rho^{1/4}$, and $\|(\mathrm{id} - \mathscr{L}_{s,\boldsymbol{t}}^{2n})^{-1}\|_{1,\tau} \ll 1$. Since we also have the crude bound $\|\mathscr{L}_{s,\boldsymbol{t}}^r\|_{1,\tau} \ll e^{O(r|1-\sigma|)}$ by (7.4) and Lemma 7.11, we deduce

$$\|(\mathrm{id} - \mathscr{L}_{s,\boldsymbol{t}})^{-1}\|_{1,\tau} \leq \|(\mathrm{id} - \mathscr{L}_{s,\boldsymbol{t}}^{2n})^{-1}\|_{1,\tau}\sum_{r=0}^{2n-1}\|\mathscr{L}_{s,\boldsymbol{t}}^r\|_{1,\tau} \ll ne^{O(n|1-\sigma|)} \ll |\tau|^{O(|\sigma-1|)}\log|\tau|,$$

Since $\|\|_\kappa$ and $\|\|_{1,\tau}$ are equivalent we obtain that the operator $\mathscr{L}_{s,\boldsymbol{t}}$ acting on H^κ has spectral radius strictly less than 1, uniformly in $|\tau| \geq \tau_1$. This concludes the proof. □

7.5. **Spectral properties of the perturbed operator.** We are now ready to generalize Proposition 7.3 to $\mathscr{L}_{s,\boldsymbol{t}}$ with arbitrary $s, \boldsymbol{t}$. We recall that we are assuming Hypothesis (M_K) and (7.1).

Proposition 7.15. *For all $s \in \mathbb{C}$ with $\mathfrak{Re}(s) \geq \sigma_0$ and $\boldsymbol{t} \in \mathbb{R}^d$, the operator $\mathscr{L}_{s,\boldsymbol{t}}$ acting on H^κ satisfies the following.*

(i) For some neighborhood U of $(1, 0)$ and all $(s, \boldsymbol{t}) \in U$, the operator $\mathscr{L}_{s,\boldsymbol{t}} : H^\kappa \to H^\kappa$ has a simple maximal eigenvalue $\lambda_{s,\boldsymbol{t}}$ and the rest of the spectrum is contained in a disk of radius strictly smaller than $\inf_{(s,\boldsymbol{t})\in U}|\lambda_{s,\boldsymbol{t}}|$.

(ii) For all $\tau_0 > 0$*, there exists* $t_0, \delta > 0$*,* $\rho \in (0,1)$ *such that* $\operatorname{spr}(\mathscr{L}_{s,\boldsymbol{t}}) < \rho$ *for* $\mathfrak{Re}(s) > 1 - \delta$*,* $|\mathfrak{Im}(s)| \geq \tau_0$*, and* $|\boldsymbol{t}| < t_0$*.*

Proof. The first item is a consequence of Proposition 7.3, equation (7.7) and perturbation theory [Kat95, Theorem IV.3.16].

For the second item, first assume that $\mathfrak{Re}(s) = 1$ and $\boldsymbol{t} = 0$. We already know that $\operatorname{spr}(\mathscr{L}_{s,0}) \leq 1$ by Proposition 7.3 and positivity. Assume by contradiction that $\operatorname{spr}(\mathscr{L}_{s,\boldsymbol{t}}) = 1$, which by the quasi-compactness in (i), means that there is an eigenvalue λ of $\mathscr{L}_{s,\boldsymbol{t}}$ of modulus 1. Let f be an eigenfunction for λ, and $\widetilde{f} := f/\xi$. Multiplying f by a constant if necessary, and by the compactness of K_Θ, we may find $(x_0, \vartheta_0) \in K_\Theta$ such that $\|\widetilde{f}\|_\infty = |\widetilde{f}(x_0, \vartheta_0)| = 1$. We first assume $\boldsymbol{t} = 0$, $\mathfrak{Re}(s) = 1$ and write $s = 1 + i\tau$, with $\tau \in \mathbb{R}_{\neq 0}$. Following the arguments in [BV05b, p. 365], we deduce that $|\widetilde{f}| = 1$ identically on K_Θ, and that for all $h \in \mathcal{H}$ and all $(y, \vartheta_1) \in K_\Theta$, we have

$$\frac{\lambda \widetilde{f}(y, \vartheta_1)}{\widetilde{f}(h(y), \vartheta_1 \vartheta_h(y))} = |h'(y)|^{i\tau}. \tag{7.21}$$

Let $b_1 \cdots b_r \in \Sigma^*$ be any cuspidal cycle such that $y \in [b_1]$. Let $c \in \Sigma$ be such that $[c, b_1] \subset K$. Finally, let $N \geq 1$ be the order of Θ, and $w := (b_1 \cdots b_r)^N$. For all $\ell \geq 1$, the word cw^ℓ belongs to $\Lambda_K(1)$, and defines a branch $h = h_{cw^\ell,1} \in \mathcal{H}$ for which $y \in \operatorname{Dom}(h)$. Using the cocycle property (5.3), we have

$$\vartheta_h(y) = \vartheta_{b_1 \cdots b_r}(b_1)^{\ell N} \vartheta_c(b_1) = \vartheta_c(b_1)$$

which is independent of ℓ. On the other hand, as $\ell \to \infty$, we have

$$h(y) = g_c g_w^\ell y \to [c, w, w, \dots] \in K.$$

By continuity, we deduce that the left-hand side of (7.21) converges as $\ell \to \infty$. On the other hand, by Lemma 4.5.(3), we have $h'(y) = g_c'(g_w^\ell y) g_{w^\ell}'(y) \sim C\ell^{-2}$ for some constant C as $\ell \to \infty$. Therefore the right-hand side of (7.21) cannot converge to a constant, yielding the desired contradiction. Thus, $\operatorname{spr}(\mathscr{L}_{1+i\tau,0}) < 1$ for $\tau \neq 0$.

By perturbation theory [Kat95, Theorems IV.3.1 and IV.2.14] and compactness, for arbitrary fixed $\tau_1 > 0$, we deduce $\operatorname{spr}(\mathscr{L}_{s,\boldsymbol{t}}) < \rho$ for $\mathfrak{Re}(s) > 1 - \delta$, $\tau_0 \leq |\mathfrak{Im}(s)| \leq \tau_1$, $|\boldsymbol{t}| < t_0$ for some $\rho \in (0,1)$ and $t_0, \delta > 0$ small enough. We pick τ_1 as in Proposition 7.14; this Proposition then ensures that the inequality $\operatorname{spr}(\mathscr{L}_{s,\boldsymbol{t}}) < \rho$ persists for all $\mathfrak{Im}(s) \geq \tau_1$, which concludes the proof. □

8. Asymptotic behaviour of the leading eigenvalue

In this section, we study the behaviour of the leading eigenvalue $\lambda(s, \boldsymbol{t}) := \lambda_{s,\boldsymbol{t}}$ of $\mathscr{L}_{s,\boldsymbol{t}}$ for $(s, \boldsymbol{t}) \approx (1, 0)$. Some of our estimates will be given also in a different form under the assumption (M_c).

The following variant of Kac's formula [PY98, Theorem 9.3] (see also [PT21, eq. (3.1)]) relates integrals relative to $\mathcal{F}_c$ to integrals relative to $\mathcal{F}_K$. We recall the definition (3.13) of the return time of $\mathcal{F}_c$ to K.

Lemma 8.1. *For* $f \in L^1(\partial\mathbb{D})$ *we have*

$$\int_K \sum_{m=0}^{r_x - 1} f(\mathcal{F}_c^m(x)) \xi(x) \, \mathrm{d}s(x) = \int_{\partial\mathbb{D}} f(x) \varphi(x) \, \mathrm{d}s(x)$$

Proof. Let $n \in \mathbb{N}$. We recall that the full measure subset $\mathcal{D} \subset \partial\mathbb{D}$ and the return map r_x are defined in Section 3.1 and (3.13) respectively. We write $K_n = \{x \in K \cap \mathcal{D} \mid r_x = n\}$ and $K_n^{>} = \{x \in K \cap \mathcal{D} \mid r_x > n\} = \{x \in K \cap \mathcal{D} \mid \mathcal{F}_c^i(x) \notin K, i = 1, \dots, n\}$ we have

$$\int_K \sum_{m=0}^{r_x-1} f(\mathcal{F}_c^m(x))\xi(x)\,\mathrm{d}s(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} \int_{K_n} f(\mathcal{F}_c^m(x))\xi(x)\,\mathrm{d}s(x) = \sum_{m=0}^{\infty} \int_{K_m^{>}} f(\mathcal{F}_c^m(x))\xi(x)\,\mathrm{d}s(x).$$

The contribution of $m = 0$ is $\int_K f(x)\xi(x)\,\mathrm{d}s(x) = \int_K f(x)\varphi(x)\,\mathrm{d}s(x)$ since $\xi(x) = \varphi(x)$ on K. Now for $m \geq 1$ we have

$$K_m^{>} = (K \cap \mathcal{D}) \cap \bigcup_{h=h_1\circ\cdots\circ h_m}^{*} h(\mathrm{Dom}(h))$$

where $*$ indicates the condition that $h_1, \dots, h_m \in \mathcal{H}_c^{\sharp}$ are such that $\mathrm{Im}(h_{j+1}) \subseteq \mathrm{Dom}(h_j)$ for $j = 1, \dots, m-1$. In particular, $\mathrm{Dom}(h) \subset \mathrm{Dom}(h_m) \subset \partial\mathbb{D} \smallsetminus K$. Thus,

$$\begin{aligned}\sum_{m\geq 1} \int_{K_m^{>}} f(\mathcal{F}_c^m(x))\xi(x)\,\mathrm{d}s(x) &= \sideset{}{^*}\sum_{h=h_1\circ\cdots\circ h_m \in \mathcal{H}_c^{\sharp},\, m\geq 1} \int_{\mathrm{Dom}(h)\cap h^{-1}(K)} f(y)\xi(h(y))|h'(y)|\,\mathrm{d}s(y) \\ &= \int_{\partial\mathbb{D}\smallsetminus K} f(y) \sideset{}{^*}\sum_{\substack{h=h_1\circ\cdots\circ h_m,\, m\geq 1\\ y\in\mathrm{Dom}(h),\, h(y)\in K}} \xi(h(y))|h'(y)|\,\mathrm{d}s(y).\end{aligned}$$

The result then follows provided we prove

$$\sideset{}{^*}\sum_{\substack{h=h_1\circ\cdots\circ h_m,\, m\geq 1\\ y\in\mathrm{Dom}(h),\, h(y)\in K}} \xi(h(y))|h'(y)| = \varphi(y) \qquad (y \in \partial\mathbb{D} \smallsetminus K). \tag{8.1}$$

Write $y \in [a] \smallsetminus K$ for some $a \in \Sigma$. By (3.18) the contribution of $m = 1$ to the left hand side is

$$\sideset{}{^*}\sum_{b\in\Sigma,\, |\widehat{a}-b|>1} \xi(h_b(y))|h_b{}'(y)| = \int_{t\in\partial\mathbb{D}\smallsetminus([a]\cup[a+1]\cup[a-1])} \frac{\zeta(\mathrm{d}y, \mathrm{d}t)}{\mathrm{d}s(y)}.$$

Now assume $m > 1$ and thus $y \in [a, \widehat{a} + \delta]$ with $\delta \in \{\pm 1\}$. If we write h as $h = h_w$ for $w = a_1 \cdots a_k$ with $w \in \Sigma^*$, then the condition $h(y) \in K$ is equivalent to $[a_1, a_2] \in K$ (i.e. $[\widehat{a}_2, \widehat{a}_1] \subseteq K$), the condition $y \in \mathrm{Dom}(h)$ is equivalent to $\widehat{a}_k = a + \delta$ and, given the other conditions, the condition $h_i \in \mathcal{H}_c^{\sharp}$ for all i and $m \geq 2$ is equivalent to $\ell_K(a_2 \cdots a_k) = 1$ and $k \geq 2$. Thus, by (3.18)

$$\begin{aligned}\sideset{}{^*}\sum_{\substack{h=h_1\circ\cdots\circ h_m,\, m\geq 2\\ y\in\mathrm{Dom}(h),\, h(y)\in K}} \xi(h(y))|h'(y)| &= \sum_{\substack{w=a_1\cdots a_k\in\Sigma^*,\, k\geq 2\\ \widehat{a}_k=a+\delta,\, [\widehat{a}_2,\widehat{a}_1]\subseteq K,\\ \ell(a_2\ldots a_k)=1}} \xi(h_w(y))|h_w'(y)| \\ &= \sum_{\substack{w=a_1\cdots a_k\in\Sigma^*,\, k\geq 2\\ \widehat{a}_k=a+\delta,\, [\widehat{a}_2,\widehat{a}_1]\subseteq K,\\ \ell(a_2\ldots a_k)=1}} \int_{t\in[\widehat{a}_k,\ldots,\widehat{a}_1]} \frac{\zeta(\mathrm{d}y, \mathrm{d}t)}{\mathrm{d}s(y)} \\ &= \int_{t\in[a+\delta]} \frac{\zeta(\mathrm{d}y, \mathrm{d}t)}{\mathrm{d}s(y)}\end{aligned}$$

and (8.1) follows by the definition (3.19) of φ. $\square$

8.1. **Perturbation theory and existence.**

Lemma 8.2. *There exist $\delta, t_0 > 0$ such that whenever $|s-1| \le \delta$ and $|\boldsymbol{t}| \le t_0$, we have*

$$\partial_{10}\lambda(s,\boldsymbol{t}) = -\mathfrak{d} + O(|s-1| + |\boldsymbol{t}| + |\boldsymbol{t}|^{\alpha-\varepsilon}), \tag{8.2}$$

where

$$\mathfrak{d} := \int_K \log|\mathcal{F}_K'(y)|\xi(y)\,\mathrm{d}s(y) = \int_{\partial\mathbb{D}} \log|\mathcal{F}_c'(y)|\varphi(y)\,\mathrm{d}s(y) > 0. \tag{8.3}$$

Proof. Proceeding as in [BD22, Lemma 7.1] and keeping track of the error terms using (7.7), we obtain the estimate (8.2) with

$$\mathfrak{d} = -\int_K \partial_{1,0}\mathscr{L}_{1,0}[\xi]\,\mathrm{d}s(y) = -\int_K \sum_{h\in\mathcal{H}} |h'(y)|\log|h'(y)|\xi(h(y))\,\mathrm{d}s(y) = \int_K \log|\mathcal{F}_K'(y)|\xi(y)\,\mathrm{d}s(y)$$

where the last equality follows by $h'(y) = 1/\mathcal{F}_K'(h(y))$ and (5.8).

Notice that for all $m \ge 1$ by the chain rule and (5.8) we have

$$\begin{aligned}\frac{1}{m}\int_K \log|(\mathcal{F}_K^m)'(y)|\xi(y)\,\mathrm{d}s(y) &= \frac{1}{m}\sum_{j=0}^{m-1}\int_K \log\left|\mathcal{F}_K'(\mathcal{F}_K^{m-j}(y))\right|\xi(y)\,\mathrm{d}s(y)\\ &= \int_K \log|\mathcal{F}_K'(y)|\xi(y)\,\mathrm{d}s(y) = \mathfrak{d}.\end{aligned}$$

The fact that $\mathfrak{d} > 0$ follows from the left-hand side expression, since by (4.5) we have $|(\mathcal{F}_K^m)'| > 1$ for m large enough. Finally, the second expression for $\mathfrak{d}$ in (8.3) follows by Lemma 8.1 since $\log|\mathcal{F}_K'(y)| = \sum_{m=0}^{r_x-1}\log|\mathcal{F}_c'(\mathcal{F}_c^m(y))|$. ☐

By the implicit function theorem (cf. [BD22, Lemma 7.2]) we obtain the following.

Lemma 8.3. *For all $\eta > 0$, there exists $t_0 > 0$ and a unique function $s_0 : [-t_0, t_0]^d \to \mathbb{C}$ such that $s_0(0) = 1$ and, for $|\boldsymbol{t}| \le t_0$,*

$$|s_0(\boldsymbol{t}) - 1| \le \eta, \qquad \lambda(s_0(\boldsymbol{t}), \boldsymbol{t}) = 1.$$

Proof. This follows from a general form of the implicit functions theorem, *e.g.* [Kum80, Theorem 1.1], whose hypotheses are satisfied by virtue of Lemma 8.2. ☐

8.2. **The sub-CLT case.** Let

$$\begin{aligned}\mathcal{J}_K &:= \int_{K_\Theta} \left(\mathrm{e}^{i\langle t, v\Phi_K(y)\rangle} - 1\right)\xi(y)\,\mathrm{d}s(y)\,\mathrm{d}v,\\ \mathcal{J}_c &:= \int_{\partial\mathbb{D}_\Theta} \left(\mathrm{e}^{i\langle t, v\Phi_c(y)\rangle} - 1\right)\varphi(y)\,\mathrm{d}s(y)\,\mathrm{d}v, \qquad\qquad \text{(assuming } (M_c))\end{aligned}$$

where Φ_K, Φ_c are defined in (5.9) and (6.8) respectively.

Lemma 8.4. *(1) For $|\boldsymbol{t}| \le t_0$, we have*

$$s_0(\boldsymbol{t}) - 1 = \frac{1}{\mathfrak{d}}\mathcal{J}_K + O(|\boldsymbol{t}|^2 + |\boldsymbol{t}|^{2\alpha-2\varepsilon}). \tag{8.4}$$

(2) If we assume (M_c) and

$$\sum_{h_1,h_2,h_3\in\mathcal{H}_c} \|h_1'\|_\infty \|h_2'\|_\infty \|h_3'\|_\infty \|\boldsymbol{\phi}_{h_1}|_{\mathrm{Im}(h_2)\cap\mathrm{Dom}(h_1)}\|_\infty \|\boldsymbol{\phi}_{h_2}|_{\mathrm{Im}(h_3)\cap\mathrm{Dom}(h_2)}\|_\infty < \infty, \tag{8.5}$$

then the same asymptotic formula holds with $\mathcal{J}_c$ instead of $\mathcal{J}_K$.

Proof. We proceed as in [BD22, Lemma 7.3] and obtain by (6.5) and (7.7)

$$\lambda(s,\boldsymbol{t}) = -\mathfrak{d}(s-1) + \int_{K_\Theta} \mathscr{L}_{1,\boldsymbol{t}}[\xi]\,\mathrm{d}\nu + O(|s-1|^2 + |\boldsymbol{t}|^2 + |\boldsymbol{t}|^{2\alpha-2\varepsilon}). \tag{8.6}$$

Noting that $\mathscr{L}_{1,\boldsymbol{t}}[\xi] = \mathscr{L}[\mathrm{e}^{i\langle \boldsymbol{t},\Phi_K\rangle}\xi]$ and using (5.8), we obtain by Hypothesis (M_K) that

$$\lambda(s,\boldsymbol{t}) = -\mathfrak{d}(s-1) + \mathcal{J}_K + O(|s-1| + |\boldsymbol{t}|^2 + |\boldsymbol{t}|^{2\alpha-2\varepsilon}). \tag{8.7}$$

Assume next Hypothesis (M_c). We start from (8.6). By Lemma 7.8 applied with $f=\xi$ we obtain

$$\int_{K_\Theta} (\mathscr{L}_{1,\boldsymbol{t}} - \mathscr{L})[\xi]\,\mathrm{d}\nu = \widetilde{\mathcal{J}}_c + \widetilde{\mathcal{I}}_c + O(|\boldsymbol{t}|^2 + |\boldsymbol{t}|^{2\alpha-2\varepsilon}).$$

where

$$\widetilde{\mathcal{J}}_c = \int_{K_\Theta} \big((\mathscr{C}_{1,\boldsymbol{t}} - \mathscr{C}) + \mathscr{C}(\mathrm{Id} - \mathscr{C}^\sharp)^{-1}(\mathscr{C}^\sharp_{1,\boldsymbol{t}} - \mathscr{C}^\sharp)\big)(\mathrm{Id} - \mathscr{C}^\sharp)^{-1}[\xi]\,\mathrm{d}\nu$$
$$\widetilde{\mathcal{I}}_c = \int_{K_\Theta} \big((\mathscr{C}_{1,\boldsymbol{t}} - \mathscr{C}) + \mathscr{C}(\mathrm{Id} - \mathscr{C}^\sharp)^{-1}(\mathscr{C}^\sharp_{1,\boldsymbol{t}} - \mathscr{C}^\sharp)\big)(\mathscr{C}^\sharp_{1,\boldsymbol{t}} - \mathscr{C}^\sharp)(\mathrm{Id} - \mathscr{C}^\sharp)^{-1}[\xi]\,\mathrm{d}\nu$$

We start by analyzing $\widetilde{\mathcal{J}}_c$. Expanding the definitions of the transfer operators, the integrand in the right-hand side becomes

$$\sum_{\substack{h=h_1\circ\dots\circ h_m, m\geq 1\\ y\in\mathrm{Dom}(h_m)}} |h'(y)| \sum_{r=1}^m \Big(\mathrm{e}^{i\big\langle \boldsymbol{t}, \vartheta_1\vartheta_{h_{r+1}\circ\dots\circ h_m}(y)\boldsymbol{\phi}_{h_r}(h_{r+1}\circ\dots\circ h_m(y))\big\rangle} - 1\Big)\xi(h(y))$$
$$= \sum_{\substack{h\in\mathcal{H}\\ y\in\mathrm{Dom}(h)}} |h'(y)| \sum_{j=0}^{r_h-1} \Big(e^{i\big\langle \boldsymbol{t}, \vartheta_1\vartheta_2(h,j,y)\Phi_c(\mathcal{F}_c^j(h(y)))\big\rangle)} - 1\Big)\xi(h(y))$$

where the sum on the first line is over $h_1,\dots,h_m$ with $h_1,\dots,h_{m-1}\in\mathcal{H}_c^\sharp$ and $h_m\in\mathcal{H}_c$ (and $\mathrm{Im}(h_i)\subseteq\mathrm{Dom}(h_{i+1})$), so that by Lemma 3.3 h ranges over $\mathcal{H}$ with $m=r_h$. In the second line, we denoted $\vartheta_2(h,j,y) = \vartheta_{h_{j+1}\circ\dots\circ h_m}(y)$. Upon integration over K_Θ, the ϑ_1-integral absorbs the factor $\vartheta_2(h,j,y)$, and we then obtain by Lemma 8.1 that $\widetilde{\mathcal{J}}_c = \mathcal{J}_c$. In the same way, the integrand in $\widetilde{\mathcal{I}}_c$ is

$$\sum_{\substack{h\in\mathcal{H}\\ y\in\mathrm{Dom}(h)}} |h'(y)| \sum_{j=0}^{r_h-2} \Big(e^{i\big\langle \boldsymbol{t}, \vartheta_1\vartheta_2(h,j,y)\Phi_c(\mathcal{F}_c^j(h(y)))\big\rangle)} - 1\Big)\Big(e^{i\big\langle \boldsymbol{t}, \vartheta_1\vartheta_2(h,j+1,y)\Phi_c(\mathcal{F}_c^{j+1}(h(y)))\big\rangle)} - 1\Big)\xi(h(y)).$$

Expressing back $h = h_1\circ\dots\circ h_m$ and passing to absolute values, this is bounded by

$$\ll |\boldsymbol{t}|^2 \sum_{\substack{h=h_1\circ\dots\circ h_m\\ m\geq 2\\ y\in\mathrm{Dom}(h)}} |h'(y)| \sum_{r=1}^{m-1} \|\boldsymbol{\phi}_{h_r}\|_{\infty,\mathrm{Im}\,h_{r+1}} \|\boldsymbol{\phi}_{h_{r+1}}\|_{\infty,\mathrm{Im}\,h_{r+2}}$$
$$\ll |\boldsymbol{t}|^2 \sum_{h_1,h_2,h_3\in\mathcal{H}_c} \|h_1\|_\infty \|h_2\|_\infty \|h_3\|_\infty \|\boldsymbol{\phi}_{h_1}\|_{\infty,\mathrm{Im}\,h_2} \|\boldsymbol{\phi}_{h_2}\|_{\infty,\mathrm{Im}\,h_3}$$

where $\operatorname{Im} h_{r+2}$ in the first line and $\operatorname{Im} h_3$ in the second line are replaced by K if $r = m-1$; here we used the boundedness of

$$\sum_{j\geq 0} \sum_{\substack{h=h_1\circ\cdots\circ h_j \\ h_j\in\mathcal{H}_c^\sharp}} \|h'\|_\infty$$

which follows from 5.3. Thus the hypothesis (8.5) readily implies that $\widetilde{\mathcal{I}}_c \ll |\boldsymbol{t}|^2$ as claimed.

Since $\mathcal{J}_K, \mathcal{J}_c, \mathcal{I}_c \ll |\boldsymbol{t}|^{\alpha-\varepsilon} + |\boldsymbol{t}|$ we deduce $s_0(\boldsymbol{t}) - 1 \ll |\boldsymbol{t}|^{\alpha-\varepsilon} + |\boldsymbol{t}|$ in either case (M_K) or case (M_c) and (8.5). Setting $s = s_0(\boldsymbol{t})$ in (8.6) we then deduce the claimed result. $\square$

Corollary 8.5. *Let $\alpha \geq 1$, then for all $\varepsilon > 0$ and all $|\boldsymbol{t}| \leq t_0$, we have*

$$s_0(\boldsymbol{t}) - 1 = i\langle \boldsymbol{t}, \mu_\Phi\rangle + O(|\boldsymbol{t}|^2 + |\boldsymbol{t}|^\alpha).$$

where

$$\mu_\Phi := \frac{\mathbf{1}_{|\Theta|=1}}{\mathfrak{d}} \times \begin{cases} \displaystyle\int_K \Phi_K(x)\xi(x)\,\mathrm{d}s(x), \\ \displaystyle\int_{\partial\mathbb{D}} \Phi_c(x)\varphi(x)\,\mathrm{d}s(x), & (\textit{assuming } (M_c) \textit{ and } (8.5)). \end{cases} \tag{8.8}$$

Proof. This follows immediately from the Taylor expansion $\mathrm{e}^{iu} = 1 + iu + O(u^2 + u^\alpha)$ and the bound (6.5). Notice that the two expressions for μ_Φ coincide by (6.7) and Lemma 8.1. $\square$

8.3. **The CLT case.** In this section we assume

(8.9) Hypothesis (M_K) holds with the additional condition $\alpha > 2$.

Our goal is to extract the term of order $|\boldsymbol{t}|^2$ in the analysis above, following [BD22, Section 7.2]. In order to do so, we introduce the following notation:

$$\begin{aligned}
\mathscr{K}[f] &:= \frac{1}{\xi}(\operatorname{Id} - \mathscr{N})^{-1}\mathscr{N}[f\xi], \\
\boldsymbol{\psi} &:= \Phi_K - \mu_\Phi \log|\mathcal{F}_K'|, \\
\boldsymbol{\chi} &:= \mathscr{K}[\boldsymbol{\psi}],
\end{aligned}$$

where $\mathscr{N}$ is as in Proposition 7.3. To see that $\boldsymbol{\chi}$ is well-defined, note first that $\mathscr{P}[\boldsymbol{\chi}] = 0$ by construction, so that

$$\begin{aligned}
\mathscr{N}[\boldsymbol{\psi}\xi](y,\vartheta) &= \mathscr{L}[\boldsymbol{\psi}\xi](y,\vartheta) \\
&= \sum_{h\in\mathcal{H}} |h'(y)|\boldsymbol{\psi}(h(y))\xi(h(y)) \\
&= \sum_{h\in\mathcal{H}} |h'(y)|\xi(h(y))\big(\Phi_K(h(y), \vartheta\vartheta_h(y)) - \mu_\Phi \log|\mathcal{F}_K'(h(y))|\big) \\
&= \sum_{h\in\mathcal{H}} |h'(y)|\xi(h(y))\big(\vartheta\boldsymbol{\phi}_h(y) + \mu_\Phi \log|h'(y)|\big)
\end{aligned}$$

by (5.11). The conditions (M_K) along with Lemma 5.4 ensures that this defines a function in H^κ. Since $\operatorname{spr}(\mathscr{N}) < 1$ on H^κ by Proposition 7.3, this show that $\boldsymbol{\chi}$ is well-defined.

Lemma 8.6. *Assuming* (8.9)*, we have*

$$s_0(\boldsymbol{t}) - 1 = i\langle \boldsymbol{t}, \mu_\Phi\rangle - \tfrac{1}{2}\boldsymbol{t}^T\Sigma_\Phi\boldsymbol{t} + O(|\boldsymbol{t}|^3 + |\boldsymbol{t}|^{\alpha-\varepsilon}), \tag{8.10}$$

with

$$\Sigma_\Phi := \frac{1}{\mathfrak{d}}\int_K \big(\boldsymbol{\psi} + \boldsymbol{\chi} - \boldsymbol{\chi}\circ\mathcal{F}_K\big)\cdot\big(\boldsymbol{\psi} + \boldsymbol{\chi} - \boldsymbol{\chi}\circ\mathcal{F}_K\big)^T \xi \,\mathrm{d}\nu. \tag{8.11}$$

Remark 8.7. The remark in [BD22, p. 1408] applies to Σ_Φ: this matrix is symmetric, positive semi-definite, and definite if and only if the image set $(\boldsymbol{\psi} + \boldsymbol{\chi} - \boldsymbol{\chi}\circ\mathcal{F}_K)(K)$ span $\mathbb{R}^d$; the analogue of the expansion (7.9) *ibid.* holds as soon as $\alpha > 1$, and the matrix Σ_Φ is well-defined whenever $\alpha > 2$.

Proof. We let $s = s_0(\boldsymbol{t}) = 1 + i\langle \boldsymbol{t}, \mu_\Phi\rangle + O(|\boldsymbol{t}|^2)$ and apply Lemma 7.7

$$\begin{aligned}
\int_{K_\Theta} \mathscr{L}_{s,\boldsymbol{t}}[\xi]\,\mathrm{d}\nu &= 1 + i\langle \boldsymbol{t}, \mu_\Phi\rangle - \mathfrak{d}(s-1) - \frac{1}{2}\int_{K_\Theta} \boldsymbol{t}^T\mathscr{L}[\widetilde{\Phi}\cdot\widetilde{\Phi}^T\xi]\boldsymbol{t}\,\mathrm{d}\nu \\
&\quad + \langle \boldsymbol{t}, \mu_\Phi\rangle \int_{K_\Theta} \Big\langle \boldsymbol{t}, \mathscr{L}[\widetilde{\Phi}\log|\mathcal{F}_K'|\xi]\Big\rangle\,\mathrm{d}\nu \\
&\quad - \frac{1}{2}\langle \boldsymbol{t}, \mu_\Phi\rangle^2 \int_{K_\Theta} \mathscr{L}[\log|\mathcal{F}_K'|^2\xi]\,\mathrm{d}\nu + O(|\boldsymbol{t}|^3 + |\boldsymbol{t}|^{1+\alpha-\varepsilon}) \\
&= 1 + i\langle \boldsymbol{t}, \mu_\Phi\rangle - \mathfrak{d}(s-1) - \frac{1}{2}\boldsymbol{t}^T \int_{K_\Theta} \boldsymbol{\psi}\cdot\boldsymbol{\psi}^T\xi\,\mathrm{d}\nu\cdot\boldsymbol{t} \\
&\quad + O(|\boldsymbol{t}|^3 + |\boldsymbol{t}|^{1+\alpha-\varepsilon}).
\end{aligned}$$

Moreover, by Lemma 7.7, for all bounded f we also have

$$(\mathscr{L}_{s,\boldsymbol{t}} - \mathscr{L})[f] = i\langle \boldsymbol{t}, \mathscr{L}[\boldsymbol{\psi} f]\rangle + O(|\boldsymbol{t}|^2).$$

Thus,

$$\begin{aligned}
&\int_{K_\Theta} (\mathscr{L}_{s,\boldsymbol{t}} - \mathscr{L})(\mathrm{Id} - \mathscr{L})^{-1}(\mathrm{Id} - \mathscr{P})(\mathscr{L}_{s,\boldsymbol{t}} - \mathscr{L})\xi\,\mathrm{d}\nu \\
&\qquad = -\int_{K_\Theta} \langle \boldsymbol{t}, \boldsymbol{\psi}\rangle(\mathrm{Id} - \mathscr{L})^{-1}(\mathrm{Id} - \mathscr{P})\mathscr{L}[\langle \boldsymbol{t}, \boldsymbol{\psi}\rangle\xi]\,\mathrm{d}\nu + O(|\boldsymbol{t}|^3 + |\boldsymbol{t}|^{1+\alpha-\varepsilon}) \\
&\qquad = -\boldsymbol{t}^T\int_{K_\Theta} \boldsymbol{\psi}\cdot\boldsymbol{\chi}^T\,\mathrm{d}\nu\cdot\boldsymbol{t} + O(|\boldsymbol{t}|^3 + |\boldsymbol{t}|^{1+\alpha-\varepsilon})
\end{aligned}$$

since $(\mathrm{Id} - \mathscr{L})^{-1}(\mathrm{Id} - \mathscr{P})\mathscr{L}[f\xi] = \xi\mathbb{K}[f]$ by the identities $\mathscr{P}\mathscr{L} = \mathscr{P}$ and $\mathscr{L} = \mathscr{P} + \mathscr{N}$.

By the third expansion in [Klo19, Theorem 2.6] with $s = s_0(t)$, we have

$$0 = \int_{K_\Theta} (\mathscr{L}_{s,\boldsymbol{t}} - \mathscr{L})\xi\,\mathrm{d}\nu + \int_{K_\Theta} (\mathscr{L}_{s,\boldsymbol{t}} - \mathscr{L})(\mathrm{Id} - \mathscr{L})^{-1}(\mathrm{Id} - \mathscr{P})(\mathscr{L}_{s,\boldsymbol{t}} - \mathscr{L})\xi\,\mathrm{d}\nu + O(|\boldsymbol{t}|^3)$$

and thus one obtains the approximation (8.10) for $s_0(\boldsymbol{t})$ with

$$\Sigma_\Phi = \frac{1}{\mathfrak{d}}\int_{K_\Theta} (\boldsymbol{\psi}\boldsymbol{\psi}^T + \boldsymbol{\chi}\boldsymbol{\psi}^T + \boldsymbol{\psi}\boldsymbol{\chi}^T)\xi\,\mathrm{d}\nu. \tag{8.12}$$

One then proceeds as in [BD22, Lemma 7.4] to show that Σ_Φ can be written as in (8.11). □

Remark 8.8. We believe an expression similar to (8.11), but involving $\mathcal{F}_c$ and its inverse branches, should hold under the assumptions (M_c). One way to prove this is to use the fact that $(K, \mathcal{F}_K)$ is obtained from $(\partial\mathbb{D}, \mathcal{F}_c)$ by induction on K, and follow the arguments worked out in [PT21]. However, following this proof would require quite more about the spectral properties of $\mathscr{C}$ (the issue is to show boundedness of the series (3.4) in [PT21]). An easier alternative way would be to argue that $\mathscr{L}_{s,\boldsymbol{t}}$ and $\mathscr{C}_{s,\boldsymbol{t}}$ share $\lambda(s,\boldsymbol{t})$ as leading eigenvalue and check that the arguments from this section carry over (this corresponds to the distributional point of view described in [PT21, Section 5]). In our applications, neither expression gives an advantage over the other for the computation of Σ_Φ, so we content with (8.11).

9. Meromorphic continuation of the quasi-inverse

We let $\partial\mathbb{D}'' := \partial\mathbb{D} \setminus (V \cup V')$, where V' denotes the border of K. Given $f : \partial\mathbb{D}''_\Theta \to \mathbb{C}$, let

$$
\|f\|_{(\kappa),\partial\mathbb{D}''_\Theta} := \sup_{\substack{a,b\in\Sigma,\\ b\neq\hat{a}}} \sup_{\substack{\vartheta\in\Theta,\, x,x'\in\partial\mathbb{D}''\cap[a,b]\\ x\neq x'}} \frac{|f(x,\vartheta)-f(x',\vartheta)|}{|x-x'|^\kappa}, \tag{9.1}
$$

$$
\|f\|_{\kappa,\partial\mathbb{D}''_\Theta} := \|f\|_{\infty,\partial\mathbb{D}''_\Theta} + \|f\|_{(\kappa),\partial\mathbb{D}''_\Theta}
$$

and let $H^\kappa_{\partial\mathbb{D}''_\Theta}$ be the space of those functions satisfying $\|f\|_{\kappa,\partial\mathbb{D}''_\Theta} < \infty$. Similarly to (7.19) we also define

$$
\|f\|_{1,\tau,\partial\mathbb{D}''_\Theta} := \|f\|_{\infty,\partial\mathbb{D}''_\Theta} + (1+|\tau|)^{-\kappa}\|f\|_{(\kappa),\partial\mathbb{D}''_\Theta}.
$$

For $s\in\mathbb{C}$ with $\mathfrak{Re}(s) > 1$ and $\boldsymbol{t}\in\mathbb{R}^d$, we define the following operator from $L^\infty(\partial\mathbb{D}''_\Theta)$ to $L^\infty(K_\Theta)$

$$
S_{\mathscr{L};s,\boldsymbol{t}}[f](y,\vartheta_1) := \sum_{\substack{h\in\mathcal{H}^*,\\ y\in\mathrm{Dom}(h)}} e^{i\langle \boldsymbol{t},\vartheta_1\phi_h(y)\rangle}|h'(y)|^s\cdot f(h(y),\vartheta_1\vartheta_h(y)), \qquad \forall (y,\vartheta)\in K_\Theta
$$

and define similarly $S_{\mathscr{C};s,\boldsymbol{t}} : L^\infty(\partial\mathbb{D}''_\Theta) \to L^\infty(\partial\mathbb{D}'_\Theta)$ in the same way but with the sum ranging over $h\in\mathcal{H}_c^*$.

We need perturbation estimates for $\mathscr{L}_{s,\boldsymbol{t}}$ acting on the space of functions with the larger domain $\partial\mathbb{D}''_\Theta$.

Lemma 9.1. *There exists $\delta>0$, such that for all $(s,\boldsymbol{t})\in\mathbb{C}\times\mathbb{R}^d$ with $\mathfrak{Re}(s) > 1-\delta$, we have*

(1) For all $f\in H^\kappa_{\partial\mathbb{D}''_\Theta}$ we have that $\mathscr{L}_{s,\boldsymbol{t}}[f]\in H^\kappa$ and is holomorphic in s.

(2) For all $f\in H^\kappa_{\partial\mathbb{D}_\Theta}$ and all $\boldsymbol{t}\ll 1$ we have

$$
\|\mathscr{L}_{s,\boldsymbol{t}}[f]\|_{1,|\mathfrak{Im}(s)|} \ll \|f\|_{1,|\mathfrak{Im}(s)|,\partial\mathbb{D}''_\Theta},
$$

$$
\|\mathscr{L}_{s,\boldsymbol{t}}[f]-\mathscr{L}[f]\|_\infty \ll (|s-1|+|\boldsymbol{t}|+|\boldsymbol{t}|^{\alpha-\varepsilon})\|f\|_{\infty,\partial\mathbb{D}''_\Theta}, \tag{9.2}
$$

$$
\|\mathscr{L}_{s,\boldsymbol{t}}-\mathscr{L}\|_{\kappa,\partial\mathbb{D}''_\Theta} \ll |s-1|+|\boldsymbol{t}|+|\boldsymbol{t}|^{\alpha-\varepsilon}. \tag{9.3}
$$

Proof. The proof is obtained proceeding as in the proofs of (7.4) and Lemma 7.11 (for $r=1$) and of (7.6)-(7.7). Notice that these results were proven for $\mathscr{L}_{s,\boldsymbol{t}}$ applied to functions $f\in H^\kappa$, but the same arguments carry over for $f\in H^\kappa_{\partial\mathbb{D}''_\Theta}$. □

In the prospect of future applications, notably that in [BDLb], it will be useful to prove an analogous result for $\mathscr{C}_{s,\boldsymbol{t}}$ and $\mathscr{C}^\sharp_{s,\boldsymbol{t}}$, where we recall that $\mathscr{C}^\sharp_{s,\boldsymbol{t}}$ is introduced in Section 6. Due to the double-average over inverse branches in hypothesis (M_c), it is natural to work with the norm

$$
\|f\|^*_{\infty,\partial\mathbb{D}''_\Theta} := \sum_{h\in\mathcal{H}_c} \|h'\|_\infty \|f\|_{\infty,(\partial\mathbb{D}''\cap\mathrm{Im}(h))\times\Theta},
$$

and define similarly $\|f\|^*_{(\kappa),\partial\mathbb{D}''_\Theta}$, $\|f\|^*_{\kappa,\partial\mathbb{D}''_\Theta}$, $H^{*\kappa}_{\partial\mathbb{D}''_\Theta}$ and $\|f\|^*_{1,\tau,\partial\mathbb{D}''_\Theta}$.

Lemma 9.2. *Assume* (M_c). *There exists* $\delta > 0$, *such that for all* $(s, \boldsymbol{t}) \in \mathbb{C} \times \mathbb{R}^d$ *with* $\mathfrak{Re}(s) > 1 - \delta$, *we have*

(1) *For all* $f \in H^{*\kappa}_{\partial\mathbb{D}''_\Theta}$ *we have that* $\mathscr{C}_{s,\boldsymbol{t}}[f], \mathscr{C}^\sharp_{s,\boldsymbol{t}}[f] \in H^{*\kappa}_{\partial\mathbb{D}''_\Theta}$ *and are holomorphic in* s.

(2) *There exists* $\rho \in (0,1)$, $t_0 > 0$ *such that for all* $|\boldsymbol{t}| \ll t_0$ *we have*

$$\|\mathscr{C}_{s,\boldsymbol{t}}\|^*_{1,|\mathfrak{Im}(s)|,\partial\mathbb{D}''_\Theta} \ll 1$$

$$\|(\mathscr{C}^\sharp)^r_{s,\boldsymbol{t}}\|^*_{1,|\mathfrak{Im}(s)|,\partial\mathbb{D}''_\Theta} \ll \rho^r, \quad r \geq 0. \tag{9.4}$$

Moreover, for all $f \in H^{*\kappa}_{\partial\mathbb{D}_\Theta}$

$$\|\mathscr{C}_{s,\boldsymbol{t}}[f] - \mathscr{C}_{1,0}[f]\|_\infty, \|\mathscr{C}^\sharp_{s,\boldsymbol{t}}[f] - \mathscr{C}^\sharp_{1,0}[f]\|_\infty \ll (|s-1| + |\boldsymbol{t}| + |\boldsymbol{t}|^{\alpha-\varepsilon})\|f\|^*_{\infty,\partial\mathbb{D}''_\Theta}.$$

Proof. As in the previous Lemma, the same argument used for the corresponding estimates for $\mathscr{L}_{s,\boldsymbol{t}}$ apply to $\mathscr{C}_{s,\boldsymbol{t}}$ and $\mathscr{C}^\sharp_{s,\boldsymbol{t}}$ as well, with the only delicate estimate being (9.4). To see this, we first observe that proceeding as in the proof of (7.4) using Lemma 5.3, we have

$$\|(\mathscr{C}^\sharp_{s,\boldsymbol{t}})^r\|^*_{\infty,\partial\mathbb{D}''_\Theta} \ll \rho^r, \quad r \geq 0 \tag{9.5}$$

for $\sigma_0 < \mathfrak{Re}(s) \ll 1$. Then, as in the proof of Lemma 7.11, for $\mathfrak{Re}(s) \in (1-\delta, 1)$ with δ sufficiently small, and enlarging $\rho \in (0,1)$ if needed, we have

$$\|(\mathscr{C}^\sharp_{s,\boldsymbol{t}})^r[f]\|^*_{(\kappa),\partial\mathbb{D}''_\Theta} \ll \rho^r\big((1+|\mathfrak{Im}(s)|)^\kappa\|f\|^*_{\infty,\partial\mathbb{D}''_\Theta} + \|f\|^*_{(\kappa),\partial\mathbb{D}''_\Theta}\big), \qquad f \in H^{*\kappa}_{\partial\mathbb{D}''_\Theta},\ r \geq 0.$$

Together with (9.5), the above bound implies 9.4. ☐

Proposition 9.3. *There exist* $t_0, C, \delta > 0$ *such that for* $\boldsymbol{t} \in \mathbb{R}^d$ *with* $|\boldsymbol{t}| \leq t_0$ *we have*

(1) *For all* $f \in H^\kappa_{\partial\mathbb{D}''_\Theta}$ *the map* $s \mapsto S_{\mathscr{L};s,\boldsymbol{t}}[f] \in H^\kappa_{\partial\mathbb{D}''_\Theta}$ *extends meromorphically to* $\mathfrak{Re}(s) > 1-\delta$, *with (at most) a unique simple pole at* $s = s_0(\boldsymbol{t})$, *with* $s_0(\boldsymbol{t})$ *as as described in Section 8.2. Moreover, for* $|s - s_0(\boldsymbol{t})| \gg 1$, $\mathfrak{Re}(s) > 1-\delta$ *we have*

$$\|S_{\mathscr{L};s}[f]\|_{1,|\mathfrak{Im}(s)|}, \ll (1 + |\mathfrak{Im}(s)|^C)\|f\|_{1,|\mathfrak{Im}(s)|,\partial\mathbb{D}''_\Theta}. \tag{9.6}$$

The residue $R_{\mathscr{L};\boldsymbol{t}}[f](y,\vartheta)$ *at* $s_0(\boldsymbol{t})$ *is*

$$R_{\mathscr{L};0}[f](y,\vartheta) = \frac{1}{\mathfrak{d}}\Big(\int_{\mathbb{D}_\Theta} f\,\mathrm{d}\nu\Big)\xi(y), \qquad (y,\vartheta) \in K_\Theta,$$

if $\boldsymbol{t} = 0$ *and otherwise it satisfies*

$$\|R_{\mathscr{L};\boldsymbol{t}}[f] - R_{\mathscr{L};0}[f]\|_{\infty,K_\Theta} \ll (|\boldsymbol{t}| + |\boldsymbol{t}|^{\alpha-\varepsilon})\|f\|_{\infty,\partial\mathbb{D}''_\Theta}. \tag{9.7}$$

(2) *Assume* (M_c). *For all* $f \in H^{*\kappa}_{\partial\mathbb{D}''_\Theta}$ *the map* $s \mapsto S_{\mathscr{L};s,\boldsymbol{t}}[f] \in H^{*\kappa}_{\partial\mathbb{D}''_\Theta}$ *extends meromorphically to* $\mathfrak{Re}(s) > 1-\delta$, *with (at most) a unique simple pole at* $s = s_0(\boldsymbol{t})$. *Moreover, for* $|s - s_0(\boldsymbol{t})| \gg 1$, $\mathfrak{Re}(s) > 1-\delta$ *we have*

$$\|S_{\mathscr{C};s}[f]\|^*_{1,|\mathfrak{Im}(s)|}, \ll (1 + |\mathfrak{Im}(s)|^C)\|f\|^*_{1,|\mathfrak{Im}(s)|,\partial\mathbb{D}''_\Theta}. \tag{9.8}$$

The residue $R_{\mathscr{C};\boldsymbol{t}}[f](y,\vartheta)$ *at* $s_0(\boldsymbol{t})$ *is*

$$R_{\mathscr{C};0}[f](y,\vartheta) = \frac{1}{\mathfrak{d}}\Big(\int_{\mathbb{D}_\Theta} f\,\mathrm{d}\nu\Big)\varphi(y), \qquad (y,\vartheta) \in \partial\mathbb{D}''_\Theta$$

if $\boldsymbol{t} = 0$ *and otherwise it satisfies*

$$\|R_{\mathscr{C};\boldsymbol{t}}[f] - R_{\mathscr{C};0}[f]\|^*_{\infty,K_\Theta} \ll (|\boldsymbol{t}| + |\boldsymbol{t}|^{\alpha-\varepsilon})\|f\|^*_{\infty,\partial\mathbb{D}''_\Theta}. \tag{9.9}$$

Proof. We start with (1), for which the argument is a straightforward generalization of *e.g.* [BD22, proof of Proposition 6.9]. By Lemma 9.1, we have $\mathscr{L}_{s,t}[f] \in H^\kappa$. Then, for $\mathfrak{Re}(s) > 1$ we have

$$S_{\mathscr{L};s,\boldsymbol{t}}[f] = \sum_{n\geq 0} \mathscr{L}^n_{s,t}[f] = \Big(f + (\mathrm{id} - \mathscr{L}_{s,t})^{-1}[\mathscr{L}_{s,t}[f]]\Big). \tag{9.10}$$

For $(s,\boldsymbol{t})$ in a neighborhood of $(1,0)$ with $\mathfrak{Re}(s) > 1$, proceeding as [BV05b, eq. (3.35)] or [BD22, Proposition 6.9] using Proposition 7.15 (i), we have that $\mathscr{L}_{s,\boldsymbol{t}}$ (acting on H^κ) can be decomposed as

$$\mathscr{L}_{s,t} = \lambda(s,\boldsymbol{t})\mathscr{P}_{s,\boldsymbol{t}} + \mathscr{N}_{s,\boldsymbol{t}}, \tag{9.11}$$

where $\mathscr{P}_{s,\boldsymbol{t}}, \mathscr{P}_{s,\boldsymbol{t}}$ are operators on H^κ which are analytic in s for $(s,\boldsymbol{t})$ in a neighborhood of $(1,0)$. Also, $\mathscr{P}_{s,\boldsymbol{t}}$ is of rank 1 and one has $\mathscr{P}_{s,\boldsymbol{t}}\mathscr{N}_{s,\boldsymbol{t}} = \mathscr{N}_{s,\boldsymbol{t}}\mathscr{P}_{s,\boldsymbol{t}} = 0$, $\mathscr{P}^2_{s,\boldsymbol{t}} = \mathscr{P}_{s,\boldsymbol{t}}$ and $\mathrm{spr}(\mathscr{N}_{s,\boldsymbol{t}}) < \rho$. It follows that

$$(\mathrm{id} - \mathscr{L}_{s,\boldsymbol{t}})^{-1} = \frac{\mathscr{P}_{s,\boldsymbol{t}}}{1-\lambda(s,\boldsymbol{t})} + (\mathrm{id} - \mathscr{N}_{s,\boldsymbol{t}})^{-1}.$$

Thus

$$S_{\mathscr{L};s,\boldsymbol{t}}[f] = \Big(\mathrm{id} + \frac{\mathscr{P}_{s,\boldsymbol{t}}\mathscr{L}_{s,t}}{1-\lambda(s,\boldsymbol{t})} + (\mathrm{id} - \mathscr{N}_{s,\boldsymbol{t}})^{-1}\mathscr{L}_{s,t}\Big)[f]. \tag{9.12}$$

This provides the claimed analytic continuation in s for $|\mathfrak{Im}(s)| \leq \tau_0$, $\mathfrak{Re}(s) > 1-\delta_0$, $|\boldsymbol{t}| < t_0$ for some $\tau_0, \delta_0, t_0 > 0$, with the possible simple pole as $s = s_0(\boldsymbol{t})$. Moreover, by the expression (7.3) for $\mathscr{P}_{1,0}$ and (5.8) we have

$$\mathscr{P}_{1,0}\mathscr{L}_{1,0}[f] = \xi \cdot \int_{K_\Theta} \mathscr{L}[f]\,\mathrm{d}\nu = \xi \cdot \int_{\partial\mathbb{D}_\Theta} f\,\mathrm{d}\nu. \tag{9.13}$$

and thus, by (8.2), we get the claimed expression for the residue of $S_{\mathscr{L};s,0}[f]$ as $s_0(0) = 1$.

The analytic continuation for $|\mathfrak{Im}(s)| > \tau_0$ and the bound (9.6) follow by (7.20), Proposition 7.15 (ii), and Lemma 9.1. Finally (9.9) follows from (9.2), (7.6)-(7.7) and [Klo19]. This concludes (1).

We now move to (2). By Lemma 3.4 we have

$$\sum_{m\geq 0} \mathscr{C}^m_{s,\boldsymbol{t}} = \sum_{m,\ell\geq 0} (\mathscr{C}^\sharp_{s,\boldsymbol{t}})^m \mathscr{L}^\ell_{s,\boldsymbol{t}}, \tag{9.14}$$

where $\mathscr{L}^\ell_{s,\boldsymbol{t}}$ is trivially extended to an operator on functions with domain $\partial\mathbb{D}''_\Theta$ by setting it to be zero outside of K_Θ if $\ell \geq 1$. The result then follows by (1) and Lemma 9.2, which implies $\sum_{m\geq 0}(\mathscr{C}^\sharp_{s,\boldsymbol{t}})^m$ extends analytically to $\mathfrak{Re}(s) > 1-\delta$, save for the expression for $R_{\mathscr{C},0}$. This follows since by (5.8)

$$\sum_{m,\ell\geq 0} (\mathscr{C}^\sharp_{1,0})^m \mathscr{P}_{1,0}\mathscr{L}_{1,0}[f] = \Big(\int_{\mathbb{D}_\Theta} f\,\mathrm{d}\nu\Big) \sum_{m\geq 0} (\mathscr{C}^\sharp_{1,0})^m[\xi\chi_K]$$

and by (8.1)

$$\sum_{m\geq 0} (\mathscr{C}^\sharp_{1,0})^m[\xi\chi_K] = \xi\chi_K + \sum_{m\geq 1} (\mathscr{C}^\sharp_{1,0})^m[\xi\chi_K] = \xi\chi_K + \varphi(1-\chi_K) = \varphi,$$

since $\varphi = \xi$ on K. $\square$

Remark 9.4. The implicit constants and the parameters t_0, δ in Proposition 9.3 can depend on the group G, on the choice of the fundamental domains R; for item (1), they may also depend on the values of the series (6.5)–(6.6), while for item (2) they may depend on the values of (6.9)–(6.10) (the expression for $s_0(\boldsymbol{t})$ in Lemma 8.4 may depend on (8.5)). There is no further dependence on $(\boldsymbol{\phi}_h)_h$ besides for these bounds.

Remark 9.5. Proposition 9.3 holds also in the case when $\boldsymbol{t}$ and the values of $\boldsymbol{\phi}_h$ are in $\mathbb{C}^d$ rather than in $\mathbb{R}^d$. This follows by identical arguments, requiring the hypotheses to be adapted as follows: for (1) it suffices to replace the conditions (6.5)–(6.6) of (M_K) by the stronger condition (finite exponential moments)

$$\sum_{h\in\mathcal{H}} \|h'\|_\infty (1 + \|\varpi\boldsymbol{\phi}_h\|_{(\kappa_0)}^\lambda) e^{\varpi\|\boldsymbol{\phi}_h\|_\infty} \leq C \tag{9.15}$$

for some $\lambda, \varpi, C > 0$ (see also [BV05b, eq. (2.5)]). The conclusion of Proposition 9.3 then holds, uniformly in $\boldsymbol{\phi}$ and ϖ for $|\boldsymbol{t}| < t_0\varpi$, for some $t_0 > 0$, so long as λ and C are fixed. For (2) it suffices instead to replace the conditions (6.9)-(6.10) of (M_c) with

$$\sum_{h_1,h_2\in\mathcal{H}_c} \|h_1'\|_\infty \|h_2'\|_\infty (1 + \|\varpi\boldsymbol{\phi}_{h_1}|_{\mathrm{Im}(h_2)\cap\mathrm{Dom}(h_1)}\|_{(\kappa_0)}^\lambda) e^{\varpi\|\boldsymbol{\phi}_{h_1}\|_\infty} < \infty. \tag{9.16}$$

Because the proof is essentially identical to the real case, we omit the details.

10. Meromorphic continuation of the generating series at cusps

In this section, we fix $\mathfrak{b} \in C(G)$. We let

$$\psi : \partial\mathbb{D} \to \mathbb{C}$$

be a fixed smooth map whose compact support avoids 1, and recall the definitions (1.5) and (1.6). In particular, by Lemma 2.1, we have

$$\sum_{x\in C(G)} |\psi(x)||q(x)|^{-2-\varepsilon} < \infty, \qquad (\varepsilon > 0).$$

We denote also

$$\psi_s(x,\vartheta) := \begin{cases} \left|\frac{x-1}{2}\right|^{-2s}\psi(x), & (\vartheta = 1,\ x \neq 1), \\ 0 & \text{(otherwise)}. \end{cases} \tag{10.1}$$

Finally, recalling (1.10) and the Bowen-Series map defined in Section 2.3, let

$$W = \{x \in \partial\mathbb{D}', \mathcal{B}(x) \in V\}.$$

Note that the set W is finite, and $W \subset K$.

We are interested in the complex-analytic behaviour of the series

$$\mathfrak{S}(s,\boldsymbol{t}) := \sum_{x\in G\mathfrak{b}} \psi(x)|q(x)|^{-2s} \mathrm{e}(\langle \boldsymbol{t}, \mathfrak{f}(x)\rangle), \tag{10.2}$$

where $\mathfrak{f} : C(G) \to \mathbb{R}^d$ satisfies the relation

$$\mathfrak{f}(h(y)) = \vartheta_h(y)^{-1}(\boldsymbol{\phi}_h(y) + \mathfrak{f}(y)), \qquad y \in \mathrm{Dom}(h), \tag{10.3}$$

for all $h \in \mathcal{H}$ (this is special case of (1.13)).

The following lemma expresses $\mathfrak{S}(s,\boldsymbol{t})$ in terms of the quasi-inverse $S_{\mathscr{L};s,\boldsymbol{t}}$ which we analyzed in the previous section.

Lemma 10.1. *For $\mathfrak{Re}(s) > 1$ and any $\boldsymbol{t} \in \mathbb{R}^d$, we have*

$$\begin{aligned}\mathfrak{S}(s,\boldsymbol{t}) = &\sum_{x\in G\mathfrak{b}\cap V} |q(x)|^{-2s} \mathrm{e}(\langle \boldsymbol{t}, \mathfrak{f}(x)\rangle)\psi_s(x,1) \\ &+ \sum_{\substack{x\in G\mathfrak{b}\cap W\\ \vartheta\in\Theta}} |q(x)|^{-2s}\mathrm{e}(\langle \boldsymbol{t}, \vartheta\mathfrak{f}(x)\rangle) S_{\mathscr{L};s,\boldsymbol{t}}[\psi_s](x,\vartheta).\end{aligned} \tag{10.4}$$

Proof. Define the sets

$$X_V := G\mathfrak{b}\cap V, \qquad X_W := G\mathfrak{b}\cap W, \qquad X_K := G\mathfrak{b}\cap K, \qquad X_O := G\mathfrak{b}\smallsetminus (X_V\cup X_W).$$

For any $f \in L^\infty(\partial\mathbb{D}_\Theta)$ and $* \in \{V, W, K, O\}$, we define the functional

$$T_*(f) := \sum_{(x,\vartheta)\in X_*\times\Theta} f(x,\vartheta)|\Upsilon(x)|^{-2s}\mathrm{e}(\langle \boldsymbol{t}, \vartheta\mathfrak{f}(x)\rangle).$$

We first show that

$$T_K = T_W + T_K\mathscr{L}_{s,\boldsymbol{t}} \qquad \text{on } L^\infty(K_\Theta). \tag{10.5}$$

Consider $x \in X_K$. If $\mathcal{B}(x) \in V$, then $x \in W$, and otherwise we have $\mathcal{F}_K(x) \in K$. Thus we have a splitting

$$T_K(f) = T_W(f) + T(f),$$

where

$$T(f) = \sum_{\substack{(x,\vartheta)\in X_K\times\Theta\\ \mathcal{F}_K(x)\in K}} f(x,\vartheta)|\Upsilon(x)|^{-2s}\mathrm{e}(\langle \boldsymbol{t}, \vartheta\mathfrak{f}(x)\rangle).$$

We wish to show that $T(f) = T_K(\mathscr{L}_{s,\boldsymbol{t}}[f])$ for $f \in L^\infty(K_\Theta)$. We sort the sum according to the value of $y = \mathcal{F}_K(x) \in K$. The condition $x \in G\mathfrak{b}$ is equivalent to $y \in G\mathfrak{b}$; and given y, the associated values of x are given by $h(y)$ for $h \in \mathcal{H}$ and $y \in \mathrm{Dom}(h)$. We thus get

$$T(f) = \sum_{y\in X_K}\sum_{\vartheta\in\Theta}\sum_{\substack{h\in\mathcal{H}\\ y\in\mathrm{Dom}(h)}} f(h(y),\vartheta)|\Upsilon(h(y))|^{-2s}\mathrm{e}(\langle \boldsymbol{t}, \vartheta\mathfrak{f}(h(y))\rangle).$$

Let $w \in \Lambda_K(1)$ be the word which determines $h = h_{w,1}$ in the notation (3.2), and let $g := g_w$, so that $h(y) = g_w y$. Then $|h'(y)| = |j(g,y)|^{-2}$, so that by (2.1), we get

$$|\Upsilon(h(y))| = |h'(y)|^{-1/2}|\Upsilon(y)|.$$

Recalling (10.3) and changing variables in the ϑ-sum, we then obtain

$$T(f) = \sum_{y\in X_K} |\Upsilon(y)|^{-2s}\mathrm{e}(\langle \boldsymbol{t}, \vartheta\mathfrak{f}(y)\rangle)\sum_{\vartheta\in\Theta}\sum_{\substack{h\in\mathcal{H}\\ y\in\mathrm{Dom}(h)}} |h'(y)|^s\mathrm{e}(\langle \boldsymbol{t}, \vartheta\boldsymbol{\phi}_h(y)\rangle)f(h(y),\vartheta\vartheta_h(y)).$$

We recognize the inner sum as $\mathscr{L}_{s,\boldsymbol{t}}[f](y,\vartheta)$, which gives $T(f) = T_K(\mathscr{L}_{s,\boldsymbol{t}}[f])$ as required.

We now consider $f \in L^\infty(\partial\mathbb{D}_\Theta)$; note that $\mathscr{L}_{s,\boldsymbol{t}}[f] \in L^\infty(K_\Theta)$. We observe that, since $x \in X_O$ is equivalent to $\mathcal{F}_K(x) \in K$ for $x \in G\mathfrak{b}$, arguing as above we have

$$T_O(f) = T_K(\mathscr{L}_{s,\boldsymbol{t}}[f]).$$

Recall the definition (10.2). A quick computation shows that $q(x) = \mathfrak{Im}(\Upsilon(x)) = \frac{x-1}{2i}\Upsilon(x)$, since with the notation (1.5), we have $x = \frac{\overline{\alpha+\beta}}{\alpha+\beta}$ and $\Upsilon(x) = \overline{\alpha+\beta}$. Therefore we have

$$\mathfrak{S}(s,\boldsymbol{t}) = \sum_{(x,\vartheta)\in G\mathfrak{b}\times\Theta} \psi_s(x,\vartheta)|\Upsilon(x)|^{-2s}\mathrm{e}(\langle \boldsymbol{t}, \vartheta\mathfrak{f}(x)\rangle)$$

with ψ_s defined in (10.1). We split the sum according to the partition $G\mathfrak{b} = X_V \cup X_W \cup X_O$. This yields

$$\mathfrak{S}(s, \boldsymbol{t}) = [T_V + T_W + T_O](\psi_s) = T_V(\psi_s) + T_W(\psi_s) + T_K(\mathscr{L}_{s,\boldsymbol{t}}[\psi_s]).$$

Since $1 \notin \operatorname{supp} \psi$, we have $\psi_s \in L^\infty(\partial\mathbb{D}_\Theta)$, and therefore by Proposition 5.4 we have

$$\mathscr{L}_{s,\boldsymbol{t}}[\psi_s] \in L^\infty(K_\Theta).$$

Iterating the relation (10.5), we find for all $J \geq 2$

$$T_K(\mathscr{L}_{s,\boldsymbol{t}}[\psi_s]) = T_W\Big(\sum_{j=1}^{J-1} \mathscr{L}_{s,\boldsymbol{t}}^j[\psi_s]\Big) + T_K(\mathscr{L}_{s,\boldsymbol{t}}^J[\psi_s]).$$

Since $\mathfrak{Re}(s) > 1$, by (7.4) we deduce that $\|\mathscr{L}_{s,\boldsymbol{t}}^J[\psi_s]\|_{\infty,K_\Theta} \ll \rho^J$ as $J \to \infty$. By Lemma 2.1, we deduce $T_K(\mathscr{L}_{s,\boldsymbol{t}}^J[\psi_s]) \ll \rho^J$ as $J \to \infty$, and therefore

$$\mathfrak{S}(s, \boldsymbol{t}) = T_V(\psi_s) + T_W\Big(\sum_{j\geq 0} \mathscr{L}_{s,\boldsymbol{t}}^j[\psi_s]\Big),$$

which is the claimed expression. □

By Proposition 9.3 and Lemma 10.1 we can immediately deduce the meromorphic continuation for $\mathfrak{S}(s, \boldsymbol{t})$ to the left of $\mathfrak{Re}(s) = 1$.

Proposition 10.2. *For some $t_0, \delta_0 > 0$, the following holds whenever $|\boldsymbol{t}| \leq t_0$.*

(1) The map $s \mapsto \mathfrak{S}(s, \boldsymbol{t})$, initially defined and analytic on $\mathfrak{Re}(s) > 1$, extends to a meromorphic function of s in the strip

$$\mathfrak{Re}(s) \geq 1 - \delta_0. \tag{10.6}$$

(2) The map $s \mapsto \mathfrak{S}(s, \boldsymbol{t})$ is analytic everywhere except, possibly, at a single point $s_0(\boldsymbol{t})$, with $s_0(\boldsymbol{t})$ as in Lemma 8.3, where it has (at most) a simple pole.

(3) If $\boldsymbol{t} = 0$ the possible pole at $s_0(0) = 1$ has residue

$$\frac{1}{\mathfrak{d}} \int_{\partial\mathbb{D}} \frac{4\psi(y)}{|y-1|^2} \,\mathrm{d}s(y) \cdot \sum_{\substack{x \in G\mathfrak{b} \smallsetminus V \\ \mathcal{B}(x) \in V}} |q(x)|^{-2} \xi(x) = \frac{1}{\pi \operatorname{vol}(\Gamma\backslash\mathbb{H})} \int_{\partial\mathbb{D}} \frac{\psi(y) \,\mathrm{d}s(y)}{|y-1|^2}$$

and in general the residue is equal to this expression up to $O(|\boldsymbol{t}| + |\boldsymbol{t}|^{\alpha-\varepsilon})$.

(4) For some $C > 0$ and all s in (10.6) with $|s - s_0(\boldsymbol{t})| \gg 1$, we have

$$|\mathfrak{S}(s, \boldsymbol{t})| \leq (2 + |\mathfrak{Im}(s)|)^C.$$

Proof of Proposition 10.2. We start with the expression (10.4). Since 1 is bounded away from the support of ψ by assumption, and V is a finite set, the first term on the right-hand side is an entire function of s bounded in vertical strips. Moreover, we have $\psi_s \in H^\kappa$, with $\|\psi_s\| \ll 1$ uniformly for s in vertical strips. Thus, since W is a finite set, the claim follows from Proposition 9.3. The second expression in (3) is obtained by partial summation from the definition (10.2) using Lemma 2.2. □

10.1. **Estimation of the characteristic function.** We are now ready to state and prove our main technical result, from which Theorem 1.3 will follow, and for which we explicitly recall the hypothesis we made throughout the paper.

Theorem 10.3. *Let $G \subset \mathrm{SU}(1,1)$ satisfy the conditions in Section 1.1, and R be an ideal fundamental domain as in Section 1.3. Let $\psi : \partial\mathbb{D} \to \mathbb{R}_+$ be a smooth, not identically zero test function with $1 \notin \operatorname{supp}\psi$. Let $\mathfrak{b}$ be a cusp of G, and let $q(x)$ be the associated denominator function from* (1.6)*. Let $(\vartheta_g)_{g\in G}$ be a multiplier system satisfying the conditions in Section 5.1 and $(\boldsymbol{\phi}_h)_{h\in\mathcal{H}}$ a family of maps as in Section 5.5 satisfying hypothesis (M_K). Finally, let $\mathfrak{f} : C(G) \to \mathbb{R}^d$ satisfy the relations* (10.3)*. Then the conclusions of Theorem 1.3 hold as stated with the following parameters:*

(i) the random variable Φ is given by $\Phi = \Phi_K(x,\vartheta)$ on the probability space K_Θ with measure $\xi(x)\,\mathrm{d}s(x)\,\mathrm{d}\vartheta$;
(ii) the normalizing quantity $\mathfrak{d}$ is given by (8.3)*;*
(iii) the mean-value vector μ_Φ and covariance matrix Σ_Φ are given by (8.8) *and* (8.11)*.*

For instance, we have

$$\mathbb{E}(\mathrm{e}^{i\langle \boldsymbol{t},\Phi\rangle}) = \int_{K_\Theta} \mathrm{e}^{i\langle \boldsymbol{t},\Phi_K(x,\vartheta)\rangle}\xi(x)\,\mathrm{d}s(x)\,\mathrm{d}\vartheta.$$

Proof. This is proved by the exact same arguments as [BD22, p. 1410, Proposition 8.1], which is a variant of [BV05b, pp. 370–372], using Proposition 10.2. ☐

Remark 10.4. The same considerations made in Remarks 9.4 and 9.5 for the case $\mathcal{O} = \mathscr{L}$ hold identically for Proposition 10.2 and Theorem 10.3.

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Dipartimento di Matematica, Università di Genova, via Dodecaneso 35, 16146 Genova, Italy
Email address: bettin@dima.unige.it

Laboratoire de Mathématiques Blaise Pascal, Université Clermont-Auvergne, Institut Universitaire de France, 3 place Vasarely, 63178 Aubière, France
Email address: sary.drappeau@uca.fr

Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany
Email address: jungwon@mpim-bonn.mpg.de