CENTRAL VALUES OF ADDITIVE TWISTS OF AUTOMORPHIC L-SERIES

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Abstract. We study series $L(x,s) := \sum_n \rho(n)\mathrm{e}^{2\pi i n x}/n^s$ with coefficients $\rho(n)$ given by Fourier coefficients of an automorphic form ϕ for a discrite co-finite subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$, and $x \in \mathbb{R}$.

Let $\mathfrak{b} \in \Gamma$ be representative in $\mathbb{R} \cup \{\infty\}$ of a fixed cusp of G. Ordering the orbit $\Gamma\mathfrak{b}$ by growing denominators, we prove that the values $L(x, 1/2)$ for $x \in \Gamma\mathfrak{b}$, suitably normalized, become equidistributed according to a standard complex Gaussian law, under necessary assumptions on the Laplace eigenvalue of ϕ. Previously this was known when either $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, or when ϕ as associated to a cuspidal holomorphic form.

This applies in particular to central values at $s = 1/2$ of additive twists of Maaß forms L-functions $\sum_{n\geq 1} a_n \mathrm{e}^{2\pi i n x}/n^s$, or to the Dirichlet-Theta series $\sum_{n\geq 1} \chi(n)\mathrm{e}^{2\pi i n^2 x}/n^s$.

Contents

1. Introduction

The present paper is concerned with the statistical distribution, as x varies among certain subsets of real numbers, of series of the shape

$$L(x) := \sum_{n\geq 1} a(n)\mathrm{e}^{2\pi i n x} \tag{1.1}$$

where $a(n)$ is a sequence resulting from a number-theoretic construction. For reasons explained below, we normalize the sequence so that $a(n) \simeq n^{-1/2+o(1)}$ on average, and we write

$$a(n) = \frac{\rho(n)}{n^{1/2}}.$$

This question arose historically from investigations on equidistribution modulo 1 of various sequences, and can be traced back at least to works of Fortet [For40], Kac [Kac46]

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and Salem-Zygmund [SZ47]; we refer to the survey [Kac49] and the recent paper [AB10] for references. We are interested here in cases where the sequence $\rho(n)$ arises from an arithmetic construction related to L-functions.

When $\rho(n)$ is constant or periodic, the map $L(x)$ is a linear combination of polylogarithms, and its values are distributed along the graph of a real-analytic map on $\mathbb{R} \smallsetminus \mathbb{Z}$. This is the case when $\rho(n)$ are the coefficient of a degree 1 L-function (in the sense *e.g.* of [IK04, Chapter 5]).

A natural next step is to consider coefficients $\rho(n)$ of a degree 2 L-function, for instance the Dirichlet convolution $\rho = \chi_1 * \chi_2$ of two Dirichlet characters, or the coefficients of the L-function associated to an automorphic form ϕ over a congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$ (*i.e.* $a(n)$ are Fourier coefficients of ϕ). The definition of the series (1.1) then requires more care; it must be understood as a value obtained by a process of analytic continuation, which involves an interplay between properties of x and of the form ϕ. In the two cases just mentioned, we may define it for all $x \in \mathbb{Q}$ by means of analytic continuation to the central point $s = 1/2$ of the series $L(x, s) := \sum_{n \geq 1} \rho(n) \mathrm{e}^{2\pi i n x}/n^s$, an argument originating to Hecke [Hec36] in the case of classical modular forms; see *e.g.* [DN26, Proposition 2.3].

This construction generalizes to forms ϕ which are automorphic with respect to an arbitrary discrete, cofinite group $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ with cusps. In this context, the set $C(\Gamma) \subset \mathbb{R} \cup \{\infty\}$ of fixed points of parabolic elements of Γ plays the role of rational numbers. There is also a corresponding notion of denominators $\mathrm{denom}(x)$, which we will recall in detail below; in particular there are asymptotically $Q^2/(\pi \operatorname{Vol}(\Gamma \backslash \mathbb{H}))$ elements in $C(\Gamma)/\mathbb{Z}$ with $\mathrm{denom}(x) \leq Q$. The process of analytic continuation of $L(x, s)$ is sensitive to the properties of $x \in C(\Gamma)$ as a cusp, which prompts the question of how the values $L(x)$ distribute in the complex plane as x varies.

Problem 1.1. *Fix a form ϕ as above and let $(\rho(n))$ be the Fourier coefficients of ϕ at some fixed cusp. As $Q \to \infty$, prove a central limit theorem for the values*

$$\{L(x), x \in C(\Gamma)/\mathbb{Z}, \mathrm{denom}(x) \leq Q\}.$$

This question has so far been solved in the following cases:

— When ϕ varies among weight 2 holomorphic cusp forms, the map $\phi \mapsto L_\phi(x)$ coincides with the modular symbol $\{\infty, x\}$. This family includes those forms associated with an elliptic curve over $\mathbb{Q}$, which prompted interest due to its link with the rank growth problem for elliptic curves, see [MR23, DFK04]. Petridis–Risager [PR18] resolved Problem 1.1 in this case, using a construction of perturbations of Eisenstein-Poincaré series [Gol99, Pet02]. A completely different proof was found by the third-named author and Sun [LS25], using dynamical systems. Finally Nordentoft [Nor21] showed how to solve the problem for holomorphic cusp forms of any even weight. The methods of [PR18, Nor21] work for holomorphic weight 2 cusp forms over any discrete, co-finite group $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ with cusps.

— When $\rho(n) = d(n)$, the number of divisors of n, then the value map $L(x)$ coincides with the central value $D(\frac{1}{2}, x)$ of the Esterman function [Est30], which Esterman used, in relation with the circle method, to count solutions to the determinant equation. This case corresponds to ϕ being a weight 2 Eisenstein series for $\mathrm{SL}(2, \mathbb{Z})$. The first two named authors proved in [BD22b] that Problem 1.1 holds true in this case.

— When $\rho(n)$ are the coefficients of an L-function attached to a Maaß cusp form for $\mathrm{SL}(2,\mathbb{Z})$, the second-named author and Nordentoft [DN26] showed, again, that Problem 1.1 holds true.

To summarize, the central limit theorem in Problem 1.1 has been proven in two different cases:

— The proofs in [PR18, Nor21] rely on a *holomorphicity* and *cuspidality* assumption on the holomorphic form ϕ, with essentially no assumptions on the underlying group Γ. The holomorphicity assumption plays a crucial role in both arguments, see the discussion in [DN26, Section 1.5], while the assumption of cuspidality is unavoidable both to the construction in [PR18] and for the method of moments used in [Nor21].

— The proofs in [BD22b, DN26] hold for the *modular group* $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, with essentially no assumptions on the form itself. The assumption $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ originates from the use of a theorem in [BD22b] on rational orbits of the Gauss maps, which derives from work of Baladi-Vallée [BV05].

The above results then leave open the cases where the level is greater than 1 and the form is either non-holomorphic or non-cuspidal holomorphic. The goal of this paper is to complete the solution of Problem 1.1 in full generality.

Theorem 1.2. *The central limit theorem claimed in Problem 1.1 holds true in general for automorphic forms of integer weight and mildly growing Fourier coefficients. More precisely, let Γ be discrete, cofinite, for which ∞ is a cusp, and let $\phi \neq 0$ be automorphic for Γ of integer weight, with eigenvalue λ_ϕ. Assume that either ϕ is cuspidal, or else $\lambda_\phi \geq 1/4$. Let $\rho(n)$ be the Fourier coefficients of ϕ at ∞. Then for some constant $\sigma_\phi > 0$ and $\beta_\phi \in \{0,1,3\}$, the multiset*

$$\Big\{ \frac{L(x)}{\sigma_\phi \sqrt{(\log Q)(\log\log Q)^{\beta_\phi}}}, x \in C(\Gamma), \mathrm{denom}(x) \leq Q \Big\} \tag{1.2}$$

converge to a standard complex Gaussian density $\mathscr{N}$ as $Q \to \infty$. We have

(i) $\beta_\phi = 0$ if ϕ is cuspidal,

(ii) $\beta_\phi = 1$ if ϕ is not cuspidal and[1] *$\phi(z) = O(\mathrm{e}^{d_{\mathbb{H}}(z,i)/2})$,*

(iii) $\beta_\phi = 3$ otherwise.

Moreover, when ϕ is cuspidal, the variance is given explicitely in terms of ϕ by the expression (2.8) *below.*

Our more general statement is Theorem 2.2 below in the cuspidal case, and Theorem 2.5 otherwise.

Remark 1.3. The distinction between the various cases for the parameter β_ϕ is better explained in terms of the shape of the growth of ϕ at cusps. This growth is characterized by the behaviour at infinity of maps $\phi_{\mathfrak{b}}$ of the shape (2.4). The case $\beta_\phi = 0$ arises when ϕ vanishes at all cusps. The case $\beta_\phi = 1$ arises when the maps $\phi_{\mathfrak{b}}$ grow asymptotically as $y^{1/2}$, and the case $\beta_\phi = 3$ arises when some $\phi_{\mathfrak{b}}$ behaves asymptotically as $y^{1/2}\log y$.

For non-cuspidal ϕ for which $\lambda_\phi < 1/4$, we don't expect a Gaussian limit, however we do expect that our methods could be adapted to prove convergence to a stable law with stability parameter given by the spectral parameter; see also Remark 1.8 below.

Remark 1.4. Regarding the existence of cuspidal Maaß forms for non-arithmetic groups, we refer to the surveys [Sar86, Sar90] and the computations in [FL05, Ave07, Str19].

[1]Here $d_{\mathbb{H}}$ the hyperbolic distance.

Remark 1.5. In Hecke congruence groups, there are about $q^{1+o(1)}$ parabolic fixed points x with $\operatorname{denom}(x) = q$. This raises the question of whether, in Problem 1.1, the limit law holds on the sets of fixed denominators $\operatorname{denom}(x) = q$ as $q \to \infty$. The computations in [MR23], motivated by the case of weight 2 holomorphic forms, indicate that this should be the case, however this is a much more difficult question, for which neither the dynamical methods used here nor the Goldfeld-Eisenstein series methods of [PR18, Nor21] are suitable. The best result in this direction, due to Blomer-Fouvry-Kowalski-Michel-Milićević-Sawin [BFK+23, Theorem 9.2], is the estimation of the second moment $\sum_{\operatorname{denom}(x)=q} L(x)^2$ for prime q. See also [MQ25, Pas25] for a related result for general q.

For generic groups, a natural condition could be $\operatorname{denom}(x) \in [q, q+1)$, however it is unlikely that anything relevant can be asked for fixed q. Indeed, based on numerical computations with the deformations of $\Gamma_0(5)$ considered in [FL05], we are led to believe that for generic Γ, the multiplicities $\{x \in C(\Gamma), \operatorname{denom}(x) = q\}$ remain bounded with respect to q, and thus statistical studies for individual values of q are not meaningful.

Remark 1.6. Using a convexity bound [IK04, Chapter 5.2], it is easy to prove that $L(x) = L^*(x) + O(1)$, where $L^*(x) := \sum_{n\geq 1} a(n) \mathrm{e}^{2\pi i n x} f(n/\operatorname{denom}(x)^2)$ and f is a smooth cut-off function with $f(0) = 1$. Thus, our work could equivalently be viewed as providing a central limit theorem for $L^*(x)$.

This raises the following natural question: instead of looking at $L(x, s)$ at the central point $s = \frac{1}{2}$, we could truncate the sum (1.1) at some parameter Q going to infinity, with x being chosen at random in $\mathbb{R}$. This question will be the subject of forthcoming work [BDLa].

As a particular case of our results, consider $r_2(n)$, the number of ways to write n as sums of two squares, and for $x \in \mathbb{Q}$, form the series

$$R(s, x) := \sum_{n\geq 1} \frac{r_2(n) \mathrm{e}^{2\pi i n x}}{n^s}.$$

This is initially defined on $\mathfrak{Re}(s) > 1$ and has an analytic continuation to $\mathbb{C} \smallsetminus \{1\}$.

Theorem 1.7. *For some $\sigma > 0$, the multiset*

$$\Big\{ \frac{R(1/2, x)}{\sigma \sqrt{\log Q \log\log Q}}, x \in \mathbb{Q}/\mathbb{Z}, \operatorname{denom}(x) \leq Q \Big\}$$

becomes equidistributed, as $Q \to \infty$, according to a standard complex Gaussian distribution.

This follows from our more general Theorem 2.5 below, upon writing $r_2(n) = 4(\mathbf{1} * \chi)(n)$ with χ the non-trivial character $\pmod 4$.

Remark 1.8. One may wonder more generally about the values at $s = 1/2$ of the analytic continuation of

$$R_k(s, x) := \sum_{n\geq 1} \frac{r_{2k}(n) \mathrm{e}^{2\pi i n x}}{n^{s+k/2-1}}.$$

The underlying automorphic form has eigenvalue $\lambda_\phi = \frac{k}{2}(1 - \frac{k}{2}) < 1/4$ if $k > 1$. On the basis of our arguments, we don't expect a Gaussian law to hold for the values $\{R_k(1/2, x)\}$, however our arguments suggest the convergence towards a stable law of stability parameter $2/k$, *e.g.* a Cauchy law for $k = 2$.

Problem 1.1 is relevant for non-integer weights as well, and our approach generalizes with little modification. Here we focus in particular on the case of theta series. Given an integer $N \geq 1$, a primitive character χ modulo N and $x \in \mathbb{Q}$, we define

$$L_\chi(s,x) := \sum_{n\geq 1} \chi(n) \frac{\mathrm{e}^{2\pi i n^2 x}}{n^s} \tag{1.3}$$

for $\mathfrak{Re}(s) > 1$ and by analytic continuation for $s \in \mathbb{C} \smallsetminus \{1\}$. In this case, we obtain a central limit theorem for $L_\chi(1/2, x)$ with variance of order $\log Q$, independently of whether the underlying modular form is cuspidal or not.

Theorem 1.9 (CLT for theta series)**.** *Let χ be a primitive character mod N and let $\sigma_\chi := \sqrt{\frac{\varphi(N)}{2N}}$ where φ is Euler's function. Then, the multiset*

$$\Big\{ \frac{L_\chi(1/2, x)}{\sigma_\chi \sqrt{\log Q}}, x \in \mathbb{Q}/\mathbb{Z}, \mathrm{denom}(x) \leq Q \Big\}$$

becomes equidistributed, as $Q \to \infty$, according to a standard complex Gaussian distribution.

Theorem 1.9 remains true if the character χ is replaced by any non-zero, N-periodic function $f : \mathbb{Z} \to \mathbb{C}$, provided that f is either even or odd[2]; see Remark 10.1. In this case, the variance is $\sigma_f^2 := \frac{1}{2N} \sum_{n \pmod N} |f(n)|^2$.

The Gaussian behaviour in Theorem 1.2 is reminiscent of results and conjectures on the behaviour of L-functions $\{L(\pi, \frac{1}{2})\}$ as π varies in certain families of automorphic representations [Sel92, Mon73, KS00, CFK+05, RS15]. The driving heuristics in these questions is the expected asymptotic independance of values $\{\rho(p), p \text{ prime}\}$, which should translate into a CLT for the *logarithm* $\log L(\pi, \frac{1}{2})$ due to the existence of the Euler product expansion. It is no surprise that the values (1.1), which do not possess an Euler product, behave in a different way. Our situation is instead more similar to that of the vertical distribution of the Hurwitz or periodic zeta-function with irrational parameters. For these no Euler product is available and it is conjectured they also satisfy a central limit theorem [HS25].

Remark 1.10. The relationship between the additive twist and GL(2) automorphy is ultimately due to the shape of unipotent subgroups of SL(2). It is a natural question to ask what remains of the limit law for sequences related to GL(3) automorphic forms, starting for example with the third-order divisor function $d_3(n) = \mathbf{1} * \mathbf{1} * \mathbf{1}(n)$. The additive twists

$$\sum_{n\geq 1} \frac{d_3(n)}{\sqrt{n}} \mathrm{e}^{2\pi i n x}$$

can be defined in the same way as for the usual divisor function, however contrary to our case here, they do not seem to enjoy the quantum modularity property on which our methods rely, which leaves open the question of the existence of a limiting law for the distribution of their values.

[2]The parity assumption arises because the theta functions associated with even and odd characters have different weights. However, it is likely that the methods presented in this paper can be adapted to handle the general case as well.

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2. Setting and results

2.1. **Automorphic forms.** Let $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ be a discrete, co-finite group with at least one cusp which, up to conjugation, we can assume to be at infinity and of width 1. We assume without loss of generality that $-\mathrm{id} \in \Gamma$, extending the group if necessary. Denote by $\mathbb{H}$ the Poincaré upper-half plane, and by $C(\Gamma) \subset \mathbb{P}(\mathbb{R})$ the set of parabolic fixed points of Γ, which for shortness we will refer to as *cusps* of $C(\Gamma)$ (although we do not identify Γ-equivalent points). In the following we let $k \in \mathbb{Z}$, except in Section 10 (where k will be half-integral). Let $\chi : \Gamma \to \mathbb{C}$ be a finite-order character, and assume the compatibility condition $\chi(-\mathrm{id}) = (-1)^k$. We form the multiplier

$$u_\gamma(z) = j_\gamma(z)^k \chi(\gamma), \qquad j_\gamma(z) := \frac{cz+d}{|cz+d|}, \qquad \big(\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right),\ z \neq \gamma^{-1}\infty\big). \tag{2.1}$$

To each cusp $\mathfrak{b} \in C(\Gamma)$, we associate its stabilizer $\Gamma_\mathfrak{b}$ in Γ, and a *scaling matrix* $\sigma_\mathfrak{b} \in \mathrm{SL}(2,\mathbb{R})$ satisfying

$$\sigma_\mathfrak{b}\infty = \mathfrak{b}, \qquad \Gamma_\mathfrak{b} = \pm\sigma_\mathfrak{b} \left(\begin{smallmatrix} 1 & \mathbb{Z} \\ & 1 \end{smallmatrix}\right) \sigma_\mathfrak{b}^{-1}. \tag{2.2}$$

Let ϕ be an automorphic form for Γ of weight k with multiplier χ, which means that

$$\phi : \mathbb{H} \to \mathbb{C}$$

is a smooth map satisfying the following conditions :

(H_1) For all $\gamma \in \Gamma$ and $z \in \mathbb{H}$, we have $\phi(\gamma z) = u_\gamma(z)\phi(z)$.
(H_2) For all $x \in C(\Gamma)$, we have $\phi(\sigma_x z) \ll y^{O(1)}$ as $y = \mathfrak{Im}(z) \to \infty$.
(H_3) The map ϕ is an eigenfunction of the hyperbolic Laplacian

$$\Delta_k = y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) - iky\frac{\partial}{\partial x}$$

with eigenvalue $\lambda_\phi = s_\phi(1-s_\phi) = 1/4 + t_\phi^2$ with $\mathfrak{Re}(s_\phi) \geq 1/2$ (here $s_\phi = 1/2 + it_\phi$).

These hypotheses imply the existence of a Fourier expansion at every cusp. Those cusps $\mathfrak{b} \in C(\Gamma)$ for which $\chi(\Gamma_\mathfrak{b}) = \{1\}$ are called singular or essential (see [Iwa97, eq. (2.65)] or [Pro03, eq. (8)]). For a singular cusp $\mathfrak{b}$, the Fourier expansion takes the shape

$$\begin{aligned} & j_{\sigma_{\mathfrak{b}}}(z)^{-k}\phi(\sigma_{\mathfrak{b}}z) \\ & = \phi_{\mathfrak{b}}(\mathfrak{Im}(z)) + \sum_{n\neq 0} a_{\mathfrak{b}}(n)\mathrm{e}(n\,\mathfrak{Re}(z))W_{\frac{k}{2}\operatorname{sgn}(n),it_\phi}(4\pi|n|\,\mathfrak{Im}(z)), \qquad (\mathfrak{b}\text{ singular}) \end{aligned} \tag{2.3}$$

where W_* is the Whittaker function,

$$e(z) := \mathrm{e}^{2\pi i z},$$

and

$$\phi_{\mathfrak{b}}(y) = \begin{cases} A_{\mathfrak{b}}y^{s_\phi} + B_{\mathfrak{b}}y^{1-s_\phi}, & (s_\phi \neq 1/2), \\ A_{\mathfrak{b}}y^{1/2} + B_{\mathfrak{b}}y^{1/2}\log y, & (s_\phi = 1/2). \end{cases} \tag{2.4}$$

We remark that $\phi_{\mathfrak{b}}(y)$ also depends mildly on the choice of the scaling matrix $\sigma_{\mathfrak{b}}$, as changing $\sigma_{\mathfrak{b}} \mapsto -\sigma_{\mathfrak{b}}$ sends $\phi_{\mathfrak{b}}(y) \mapsto (-1)^k\phi_{\mathfrak{b}}(y)$.

For non-singular cusps, a similar expansion holds with all terms having a uniform exponential decay as $\mathfrak{Im}(z) \to \infty$, see [Pro03, eq. (13)]. Thus, for some $c > 0$ and all non-singular cusp $\mathfrak{b}$, we have

$$\phi(\sigma_{\mathfrak{b}}z) \ll_{\mathfrak{b}} \mathrm{e}^{-c\,\mathfrak{Im}(z)}, \qquad (\mathfrak{Im}(z) \gg 1, \mathfrak{b}\text{ non-singular}).$$

We assume further the following bound on the Fourier coefficients :

(H_4) The Fourier coefficients $a_{\mathfrak{b}}(n)$ in (2.3) satisfy

$$\sum_{n\geq 1} \frac{|a_{\mathfrak{b}}(n)|}{n^{1/2+\varepsilon}} \ll_{\phi,\varepsilon} 1. \tag{2.5}$$

The hypothesis (H_4) is automatically satisfied when ϕ is cuspidal by [Iwa02, Theorem 3.2]. The same argument can be carried out when ϕ is not cuspidal and $\mathfrak{Re}(s_\phi) = \frac{1}{2}$ (*i.e.* $\lambda_\phi \geq 1/4$), using the L^2 bound $\int_{\Gamma\backslash\mathbb{H}} \left|E^Y(z)\right|^2 \mathrm{d}z \ll \log Y$ on truncated Eisenstein series in [Iwa02, eqs. (6.35)–(6.36)].

If $s_\phi \equiv \frac{k}{2} \pmod 1$, then ϕ is associated to a holomorphic modular form in the sense that $z \mapsto y^{-k/2}\phi(z)$ is holomorphic of weight k.

In the rest of the paper, the form ϕ is fixed, and apart from Section 10, we assume everywhere that either ϕ is cuspidal, or $\mathfrak{Re}(s_\phi) = \frac{1}{2}$. Our main focus is on the Fourier coefficients at ∞, which we denote

$$a(n) := a_\infty(n).$$

2.2. **Additive twists.** Given ϕ as above, we define for all $x \in C(\Gamma)$ the following L-series[3]:

$$L(\phi, s, x) := \sum_{n\geq 1} a(n)\mathrm{e}(nx)n^{1/2-s}. \tag{2.6}$$

By [DN26, Section 5], this function of s, initially defined for $\mathfrak{Re}(s) > 1$, extends to a meromorphic function on $\mathbb{C}$ which is regular at $s = 1/2$.

We define

$$L_\phi(x) := L(\phi, 1/2, x), \qquad (x \in C(\Gamma)).$$

[3]In this work we focus on the sum over positive n only; the arguments in [DN26] could be adapted with minor adjustements if one wishes to include the sum over $n \leq -1$ in the analysis.

2.3. **Orbits and denominators.** In the case $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, or more generally for congruence groups, the set of cusps is simply $C(\Gamma) = \mathbb{Q}$, and we have a natural ordering according to denominators. We recall from [PR18] the appropriate generalization for arbitrary Fuchsian groups, the role of the "denominator" being played by the bottom-left coefficient of an associated element of Γ.

Let $\mathfrak{b} \in C(\Gamma)$ be some fixed cusp, and $\sigma_{\mathfrak{b}}$ be a scaling matrix. We order the set $\Gamma\mathfrak{b}$ of cusps equivalent to $\mathfrak{b}$ according to the following definition: for all $Q > 0$, let

$$C_{\mathfrak{b}}(\Gamma; Q) := \{\gamma\mathfrak{b}, \gamma \in \Gamma/\Gamma_{\mathfrak{b}}, \gamma\sigma_{\mathfrak{b}} = \left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right), 0 < |c| \le Q\}. \tag{2.7}$$

In the notation of [PR18, eq. (1.5)], we have $C_{\mathfrak{b}}(\Gamma; Q) = T_{\infty\mathfrak{b}}(Q) + \mathbb{Z}$. Thus, for example, $C_\infty(\mathrm{SL}(2,\mathbb{Z}); Q)$ is the set of all rationals of denominator at most Q. The sets $C_{\mathfrak{b}}(\Gamma; Q)$ are all invariant under translation by $\mathbb{Z}$.

2.4. **Probability space.** We fix from now on a cusp $\mathfrak{b} \in C(\Gamma)$, and consider the space

$$\Omega_Q := C_{\mathfrak{b}}(\Gamma; Q),$$

for $Q \ge 1$.

The following lemma, which corresponds essentially to [PR18, Theorem 3.8], is borrowed from [BDLb, Lemma 2.2].

Lemma 2.1. *There exists $\delta > 0$ such that for all smooth and compactly supported $\psi : \mathbb{R} \to \mathbb{C}$, we have*

$$\sum_{x \in \Omega_Q} \psi(x) = \frac{Q^2}{\pi \operatorname{Vol}(\Gamma\backslash\mathbb{H})} \int_{\mathbb{R}} \psi(r)\,\mathrm{d}r + O_\psi(Q^{2-\delta}).$$

We fix once and for all a smooth, compactly supported, non-negative and non-identically zero weight function $\psi : \mathbb{R} \to \mathbb{R}_{\ge 0}$. By the previous lemma, for Q large enough in terms of $\mathfrak{b}$ and ψ we may endow Ω_Q with the probability measure $\mathbb{P}_Q$ proportional to ψ,

$$\mathbb{P}_Q(\delta_x) = \varpi_\psi \psi(x), \qquad (x \in \Omega_Q)$$

where $\varpi_\psi = (\sum_{x \in \Omega_Q} \psi(x))^{-1} > 0$ is a normalizing factor and δ_x is the Dirac mass at x. We call $\mathbb{E}_Q$ the associated expectation. Thus, for any map $f : C(\Gamma) \to \mathbb{C}$, we have

$$\mathbb{E}_Q[f(x)] = \frac{\sum_{x \in \Omega_Q} \psi(x) f(x)}{\sum_{x \in \Omega_Q} \psi(x)}.$$

2.5. **Distribution of values.** Our main results are the following.

Theorem 2.2 (CLT for cuspidal forms)**.** *Assume that ϕ is cuspidal. Let $\mathscr{R} \subset \mathbb{C}$ be a Borel set with measure zero boundary. As $Q \to \infty$, we have*

$$\mathbb{P}_Q\Big(\frac{L_\phi(x)}{\sigma_\phi \sqrt{\log Q}} \in \mathscr{R}\Big) \to \mathbb{P}(N_{\mathbb{C}}(\mathrm{Id}) \in \mathscr{R}),$$

where $N_{\mathbb{C}}(\mathrm{Id})$ is a standard complex Gaussian, and

$$\sigma_\phi^2 = \frac{1}{\Gamma(1 - s_\phi + \frac{k}{2})\Gamma(s_\phi + \frac{k}{2})} \frac{\|\phi\|^2}{\operatorname{Vol}(\Gamma\backslash\mathbb{H})}. \tag{2.8}$$

with $\|\phi\|$ the Petersson L^2-norm.

Remark 2.3. The proof gives a uniform and optimal rate of convergence $O(1/\sqrt{\log Q})$ when $\mathscr{R}$ varies over convex Borel sets, as in the usual multivariate CLT when the third moment is bounded, see [BRR10, Theorem 13.2].

Remark 2.4. Theorem 2.2 is derived from an estimate on the characteristic function $\mathbb{E}_Q(\mathrm{e}^{i\langle \boldsymbol{t}, L_\phi(x)\rangle})$ for $\boldsymbol{t} \in \mathbb{R}^2$. As in [BD22b, Theorem 2.4], the present method also gives large deviations estimates, using the cuspidality of ϕ. This fact was known for forms which are holomorphic of weight 2 [PR18, LS25] or of level 1 [BD22b, DN26]. The work of Nordentoft [Nor21], which considers holomorphic forms in general, relies on the method of moments, which at present cannot give a large deviation estimate.

Theorem 2.5 (CLT for non-cuspidal forms)**.** *Let ϕ be non-cuspidal with $\mathfrak{Re}(s_\phi) = \frac{1}{2}$, and denote*

$$\beta_\phi = \begin{cases} 3, & (s_\phi = \frac{1}{2} \text{ and } \exists \mathfrak{b}, B_\mathfrak{b} \neq 0), \\ 1 & (\text{otherwise}), \end{cases} \tag{2.9}$$

where we recall the notation (2.4)*. Then, there exists $\sigma_\phi > 0$ such that for any domain $\mathscr{R} \subset \mathbb{C}$, as $Q \to \infty$, we have*

$$\mathbb{P}_Q\Big(\frac{L_\phi(x)}{\sigma_\phi \sqrt{\log Q (\log\log Q)^{\beta_\phi}}} \in \mathscr{R}\Big) \to \mathbb{P}(N_\mathbb{C}(\mathrm{Id}) \in \mathscr{R}). \tag{2.10}$$

The value of σ_ϕ could be made explicit in terms of a fundamental region for Γ and certain values arising in the course of our proof, see Section 7.3.

When $\Gamma = \Gamma_0(N)$ is a Hecke congruence subgroup, given any pair (χ_1, χ_2) of primitive Dirichlet characters mod N, we may pick $\phi = E_{\chi_1,\chi_2}$ as an Eisenstein series for which $\rho(n) = \sum_{d|n} \chi_1(d)\chi_2(n/d)$. Then $\beta_\phi = 3$ if $\chi_1 = \overline{\chi_2}$ and $\beta_\phi = 1$ otherwise.

As we mentioned, the proofs of Theorems 2.2 and 2.5 are based on the asymptotic analysis of the characteristic functions, rather than the method of moments. Nonetheless, the method also yields the asymptotic estimate for all convergent moments of $L_\phi(x)$, which is to say, all moments in the cuspidal case and the first moment in the non-cuspidal case. In Section 8 we analyze in particular the expectation and the variance (in the cuspidal case) in order to determine the parameters of the central limit theorems in Theorems 2.2 and 2.5.

Remark 2.6. Given a finite family $\phi_1, \dots, \phi_n$ of orthogonal Hecke-Maass cusp forms for a group Γ and a character χ, the arguments described here can be used to obtain, more generally, the joint convergence of the vector

$$\Big(L_{\phi_j}(x)/\sigma_{\phi_j}\sqrt{\log Q}\Big)_{1\leq j\leq n} \tag{2.11}$$

to the standard n-dimensional complex Gaussian distribution, generalizing [DN26, Corollary 7.7] which dealt with the case $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. To prove this, we can follow [DN26] or, more directly, apply Theorem 2.2 to $\sum_{j=1}^n a_j L_{\phi_j}(x) = L_{\sum_{j=1}^n a_j\phi_j}(x)$ for any $a_1, \dots, a_n \in \mathbb{C}$ and invoke the Cramér–Wold device [CW36] (see [Bil95, Theorem 29.4]). Normalising suitably the components of the vector (2.11) one can also remove the cuspidality hypothesis.

When $s_\phi \not\equiv \frac{k}{2} \pmod 1$, the level-raising and lowering operators alter the coefficients $(a(n))_{n>0}$ by a non-zero constant factor (see [DFI02, eqs. (4.25)–(4.28)]. Thus, for the sake of proving Theorems 2.2 and 2.5, we may assume from now on that

$$s_\phi \not\equiv \tfrac{k}{2} \pmod 1 \implies k \in \{0, 1\}. \tag{2.12}$$

2.6. **Overview.** In the present paper, we take up the method introduced in [LS25, BD22b]. One key ingredient is the authors' work [BDLb], where Baladi–Vallée's methods are transfered to the context of Fuchsian groups of the first kind, and the role of the Gauss map is played by the Bowen-Series map [BS79].

The "observables" in our work are maps of the shape

$$h_\gamma^L(x) := L_\phi(\gamma x) - u_\gamma(x) L_\phi(x), \tag{2.13}$$

for which we have no sensible explicit expression. This poses a major issue regarding our two tasks: (i) to check that the hypotheses of [BDLb] are satisfied in our case, and (ii) to compute the parameters of the limit laws.

One of the main features in [BDLb] is a choice of induction of the Bowen-Series map whose inverse branches are easily described, as well as flexible hypothesis on the observables (existence of some positive absolute moment). Section 3 is devoted to proving that the necessary regularity and growth assumptions are satisfied by the maps (2.13) for $\gamma = \gamma_0^n$, where γ_0 is fixed, and n and x vary. This extends work of [DN26], which corresponds to the case $n = 1$. The main challenge is to achieve complete uniformity, both in n and x, and in fact, we will also need to consider the more general period function $h_\gamma^L(x, s)$ associated with $L(\phi, s, x)$, and analyze its asymptotic behaviour uniformly in the complex parameter s.

Section 4 is devoted to the study of certain Fourier integrals that arise in the analysis of the main terms, while Sections 5 and 6 introduce the setting required to apply the results of [BDLb]. Then, in Section 7, we apply the results of [BDLb], proving Theorems 2.2 and 2.5 up to the values of the parameters of the limit law. In the course of the argument, we must show that the multiplier system u_γ satisfies a certain topological mixing condition. This will be carried out by an ad-hoc combinatorial construction with Bowen–Series codings, which we defered in Section 9. At that point, we won't have an explicit expression for the expectation and the covariance matrix of the limiting distributions yet; computing the expectation and covariance matrix requires an unexpected amount of additional work.

First, in Section 7.3, we show that the Gaussian law in Theorem 2.5 is circular symmetric. This requires an analysis of various contributions to the variance, and two symmetries. The first is the involution $\gamma \mapsto \gamma^{-1}$ over parabolic elements of Γ. The second is the invariance of a suitably constructed finite index subgroup of Γ under conjugation $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix}\right)$.

In Section 8, we compute the expectations in both the cuspidal and the Eisenstein cases, and the covariance matrix in the cuspidal case. To carry this out, we introduce a perturbation argument which expresses both quantities in terms of limits, as $s \to 1/2$ with $s > 1/2$, of continuous averages of $L(\phi, s, x)$ over x. These averages are estimated using standard analysis (for the expectation) and Rankin-Selberg theory (for the variance). A more detailed overview is given in Section 8.1.

Finally, in Section 10 we consider the case of half-integral weights and prove Theorem 1.9.

Notations. The Landau symbol $f = O(g)$ means that there exists a constant $C \geq 0$ such that $|f| \leq Cg$ whenever f and g are defined. The notation $f \ll g$ is equivalent to $f = O(g)$. If the constant depends on a parameter, say α, this is typically indicated by a subscript, *e.g.* $f = O_\alpha(g)$ or $f \ll_\alpha g$. However, we will not explicitly indicate dependencies on ϕ, ε and on the group Γ, nor on the subgroup G and the fundamental domain R defined in Sections 5 and 6.

Given a matrix $\gamma \in \mathrm{SL}(2,\mathbb{C})$, we indicate by c_γ the lower-left entry of γ. Throughout the paper, the letter ε indicates a positive number which can be taken arbitrarily small, and whose value might vary from line to line. The indicator function of a condition C is denoted by $\mathbf{1}_C$, with $\mathbf{1}$ denoting the constant function 1.

3. Estimates on the period function

3.1. **Analytic continuation of the Eichler integral.** As in [DN26], we start with the Eichler integral $\mathcal{E}_x(\phi,s)$. For $s\in\mathbb{C}$ with $\mathfrak{Re}(s)>1$ and all $x\in\mathbb{R}$, we define

$$\mathcal{E}_x(\phi,s) := \int_0^\infty \big(\phi(x+iy)-\phi_\infty(y)\big)y^{s-1/2}\frac{\mathrm{d}y}{y}. \tag{3.1}$$

We will see later on, in (3.39), how to transfer informations about $\mathcal{E}_x$ to L. The maps $\mathcal{E}_x(\phi,s)$ are related to $L(\phi,s,x)$ in the same way as automorphic periods relate to L series, see [DN26, Section 5.4].

Recall our assumption that $\mathfrak{Re}(s_\phi)=1/2$ whenever ϕ is non-cuspidal. For convenience, we denote

$$\rho_\phi := \begin{cases} 0, & (\phi \text{ cuspidal}),\\ 1/2 & (\text{otherwise}),\end{cases} \tag{3.2}$$

and we recall the notation β_ϕ from Theorem 1.2.

Lemma 3.1. *We have*

$$|\phi(x+iy)-\phi_\infty(y)| \ll y^{-\rho_\phi}(1+|\log(1/y)|^{(\beta_\phi-1)/2})\mathrm{e}^{-y/3} \qquad (x\in\mathbb{R}, y>0). \tag{3.3}$$

Proof. For $y\gg 1$, this follows from the Fourier expansion (2.3), the bound (2.5) and the exponential decay of Whittaker function [GR07, 9.227].

Assume then that y is small. If ϕ is cuspidal, the right-hand side is $\asymp 1$ and thus the bound is obvious by boundedness of ϕ. Assume that ϕ is not cuspidal, let $z=x+iy$ and let F be the Ford fundamental domain [Iwa02, Chapter 2.2]. For all $Y\geq 1$, denote also $F(Y)$ the central part of F, in the sense of [Iwa02, Chapter 2.2], and $Y_0\geq 1$ be such that the cuspidal parts $F_{\mathfrak{b}}(Y_0)$ are disjoint. Let $\gamma\in\Gamma$ be such that $z':=\gamma z\in F$. If $\gamma\in\Gamma_\infty$, it follows that $\mathfrak{Im}\,\gamma z=\mathfrak{Im}\,z$. Otherwise we have on the one hand $\mathfrak{Im}\,\gamma z\geq \mathfrak{Im}\,z$ since $\gamma z\in F$, and on the other hand $\gamma=\left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)$ with $c\neq 0$ (since $\gamma\notin\Gamma_\infty$) and thus $\mathfrak{Im}\,\gamma z\leq \frac{1}{c^2\,\mathfrak{Im}\,z}\ll 1/y$. We deduce that $y\leq \mathfrak{Im}\,z'\ll 1/y$.

Let $\mathfrak{b}$ be such that $\mathfrak{Im}(\sigma_{\mathfrak{b}}^{-1}z')\gg 1$ (we pick $\mathfrak{b}=\infty$ if $z\in F(Y_0)$, and otherwise $\mathfrak{b}$ the unique cusp for which $z'\in F_{\mathfrak{b}}(Y_0)$). Then $|\phi(z)|=|\phi(z')|=|\phi(\sigma_{\mathfrak{b}}w)|$ where $w\in F$ satisfies $\mathfrak{Im}\,w\gg 1$. Arguing as above, we have $\mathfrak{Im}(w)=\mathfrak{Im}(z')\ll 1/y$ if $\mathfrak{b}=\infty$, and otherwise $\mathfrak{Im}(w)\ll 1/\mathfrak{Im}(z')\leq 1/y$ (since there are finitely many possibilities for $\mathfrak{b}$). In any case $\mathfrak{Im}(w)\ll 1/y$.

We conclude that

$$\begin{aligned} |\phi(x+iy)-\phi_\infty(y)| &\ll |\phi(z)|+1\\ &= |\phi(\sigma_{\mathfrak{b}}w)|+1\\ &\ll |\phi_{\mathfrak{b}}(\mathfrak{Im}\,w)|+1\end{aligned}$$

again by the exponential decay of the Whittaker function and the bound (2.5). Finally by the assumption $\mathfrak{Re}(s_\phi)=1/2$ and the definition of β_ϕ, we get $|\phi_{\mathfrak{b}}(\mathfrak{Im}\,w)|\ll y^{-1/2}$ if $\beta_\phi=1$, and otherwise $|\phi_{\mathfrak{b}}(\mathfrak{Im}\,w)|\ll y^{-1/2}\log(1+1/y)$ as claimed. ∎

Proceeding as in [DN26, Lemma 2.2], we also have

$$|\phi(x+iy)-\phi(x'+iy)| \ll y^{-\rho_\phi+\varepsilon}\mathrm{e}^{-y/3}\min(1,|x-x'|/y) \qquad (x,x'\in\mathbb{R}, y>0). \tag{3.4}$$

Lemma 3.2.

(i) For each $x\in\mathbb{R}$, the map $s\mapsto\mathcal{E}_x(\phi,s)$ is holomorphic on $\mathfrak{Re}(s)>\frac12+\rho_\phi$.

(ii) For $\mathfrak{Re}(s)>\frac12+\rho_\phi$, the map $x\mapsto\mathcal{E}_x(\phi,s)$ is continuous.

(iii) For each $\varepsilon>0$, we have $\mathcal{E}_x(\phi,s)\ll 1$ uniformly for $\frac12+\rho_\phi+\varepsilon\le\mathfrak{Re}(s)\ll 1$.

(iv) For $x\in C(\Gamma)\smallsetminus\{\infty\}$, the map $s\mapsto\mathcal{E}_x(\phi,s)$ extends to an analytic function on

$$\mathcal{D}_\phi := \begin{cases}\mathbb{C}, & (\phi \text{ cuspidal}),\\ \mathbb{C}\smallsetminus\{\pm it_\phi, 1\pm it_\phi\}, & (\phi \text{ non-cuspidal}).\end{cases}$$

(v) For each fixed $x\in C(\Gamma)\smallsetminus\{\infty\}$, and $s\in\mathbb{C}$ satisfying

$$\mathfrak{Re}(s)\ll 1, \qquad \operatorname{dist}(s,\mathbb{C}\smallsetminus\mathcal{D}_\phi)\gg 1,$$

we have

$$\mathcal{E}_x(\phi,s)\ll_x 1. \tag{3.5}$$

Proof. The first three properties are consequences of the bound (3.3). The fourth property is [DN26, Proposition 2.3], while the last property is a direct consequence of the uniform boundedness of [DN26, eq. (2.13)]. □

3.2. **Hölder regularity of the period function $h^{\mathcal{E}}$ at $s=1/2$.** In accordance with [DN26, eq. (3.13)], for all $x\in C(\Gamma)\smallsetminus\{\infty,\gamma^{-1}\infty\}$ and $s\in\mathcal{D}_\phi$, we let

$$h^{\mathcal{E}}_\gamma(x,s) := |j(\gamma,x)|^{2s-1}\mathcal{E}_{\gamma x}(\phi,s)-u_\gamma(x)\mathcal{E}_x(\phi,s),$$

where we recall the definition (2.1) of $u_\gamma(x)$. Note that $h^{\mathcal{E}}_\gamma(x,s)=0$ for $\gamma\in\Gamma_\infty$; in the remainder of this section we assume

$$\gamma\notin\Gamma_\infty.$$

We also notice that for $\gamma_1,\gamma_2\in\Gamma$, $h^{\mathcal{E}}$ satisfies the period relation

$$h^{\mathcal{E}}_{\gamma_1\gamma_2}(x,s)=|j(\gamma_2,x)|^{2s-1}h^{\mathcal{E}}_{\gamma_1}(\gamma_2x,s)+u_{\gamma_1}(\gamma_2x)h^{\mathcal{E}}_{\gamma_2}(x,s) \tag{3.6}$$

whenever the involved quantities are defined.

Lemma 3.3. *The map $x\mapsto h^{\mathcal{E}}_\gamma(x,\frac12)$ extends to a function on $\mathbb{R}\smallsetminus\{\gamma^{-1}\infty\}$ which is locally κ-Hölder continuous for any $0<\kappa<1-\rho_\phi$. More precisely, for any $\varepsilon>0$ we have*

$$\big|h^{\mathcal{E}}_\gamma(x,\tfrac12)-h^{\mathcal{E}}_\gamma(x',\tfrac12)\big| \ll_\gamma (1+|x|+|x'|)^{2\varepsilon}|x-x'|^{1-\rho_\phi-\varepsilon} \tag{3.7}$$

uniformly in $x,x'\in\mathbb{R}\smallsetminus[\gamma^{-1}\infty-\varepsilon,\gamma^{-1}\infty+\varepsilon]$.

Proof. This is [DN26, Theorem 3.1], except that the exponent of $(1+|x|+|x'|)$ appearing there is raised by an unspecified constant. This improved bound is already implicit in the proof in the cuspidal case (using Wilton's bound in [DN26, eq. (3.17)]), whereas for the non-cuspidal case it immediately follows from [DN26, Theorems 3.1 and 4.1] unless $|x|\to\infty$ and $|x-x'|\le|x|/2$. To handle this remaining case, one could simply import the techniques of Theorem 3.1 into the proof of Theorem 4.1 therein. An alternative method, suggested by an LLM and which is less direct but easier to implement here, is to proceed as follows.

With the notation of the proof of [DN26, Theorem 3.1], we have

$$h^{\mathcal{E}}_\gamma(x,\tfrac12)=-\Delta_0(x,\tfrac12)+\Delta_c(x,\tfrac12)-\Delta_\infty(x,\tfrac12), \tag{3.8}$$

where $\Delta_c'(x,\frac{1}{2}), \Delta_\infty'(x,\frac{1}{2}) \ll (1+|x|)$, by a trivial bound and (3.17) *ibid.* respectively. Also, $\Delta_0(x,\frac{1}{2}) = H(x)$ satisfies $\Delta_0(x,\frac{1}{2}) = O(|x|^{-1})$, by inserting the bound following (4.12) in (3.9) *ibid.* and $\Delta_0(x,\frac{1}{2}) - \Delta_0(x',\frac{1}{2}) \ll |x-x'|^{1/2-\varepsilon}$, by (3.10) *ibid.* if $|x-x'| \ll 1$ and trivially otherwise.

Furthermore, [DN26, Theorem 4.1] gives

$$h_\gamma^{\mathcal{E}}(x,\tfrac{1}{2}) = M(x) + E(x) \tag{3.9}$$

for an explicit main term M, clearly satisfying $M'(x) \ll |x|^{-1/2+\varepsilon}$ and $M(x) - M(x') \ll |x-x'|^{1/2-\varepsilon}|x|^{2\varepsilon}$, and an error term $E(x) = O(|x|^{-1+\varepsilon})$.

Combining (3.8)-(3.9), we have that $h_\gamma^{\mathcal{E}}(x,\frac{1}{2}) = M(x) - \Delta_0(x,\frac{1}{2}) + R(x)$, with

$$R(x) = E(x) + \Delta_0(x,\tfrac{1}{2}) = \Delta_c(x,\tfrac{1}{2}) - \Delta_\infty(x,\tfrac{1}{2}) - M(x).$$

In particular, it suffices to show that $R(x)$ satisfies the the claimed bound, since M and Δ_0 do. This then immediately follows since the two expressions for R give $R(x) \ll |x|^{-1+\varepsilon}$ and $R'(x) \ll |x|$ respectively, whence

$$R(x) - R(x') \ll (|R(x)| + |R(x')|)^{1/2}|R(x) - R(x')|^{1/2} \ll |x|^{-1/2+\varepsilon}\big(|x-x'|\cdot|x|\big)^{1/2},$$

which readily gives the claim. □

3.3. **Upper-bound on the period function $h^{\mathcal{E}}$.** We recall from [DN26, Lemma 4.5] the definition

$$\Upsilon_{k,\eta}(\tau) := \mathrm{e}^{\pi i(2-\eta)k/4}\frac{2\pi\Gamma(\tau)}{4^\tau\Gamma(\frac{\tau+1+k/2}{2})\Gamma(\frac{\tau+1-k/2}{2})}, \qquad (k\in\mathbb{Z}, \eta=\pm1, \tau\in\mathbb{C}\smallsetminus\mathbb{Z}_{\leq 0}), \tag{3.10}$$

which will be relevant for non-cuspidal forms. For $s\in\mathbb{C}$, $s\notin \pm(s_\phi-\frac{1}{2})+\mathbb{Z}_{\leq 0}$, and $\beta\in\mathbb{R}_{\neq 0}$, we define

$$\mathcal{M}_\beta(\phi,s) := \begin{cases} A_\infty|\beta|^{s_\phi-\frac{1}{2}+s}\Upsilon_{k,\eta}(s_\phi-\frac{1}{2}+s) + B_\infty|\beta|^{\frac{1}{2}-s_\phi+s}\Upsilon_{k,\eta}(\frac{1}{2}-s_\phi+s), & (s_\phi\neq\frac{1}{2}),\\ (A_\infty + B_\infty\log|\beta|)|\beta|^s\Upsilon_{k,\eta}(s) + B_\infty|\beta|^s\Upsilon_{k,\eta}'(s), & (s_\phi=\frac{1}{2}),\end{cases} \tag{3.11}$$

where A_∞, B_∞ are the coefficients in the expansion (2.4) at the cusp at ∞, and $\eta = \operatorname{sgn}(\beta)$. We will denote them by $A_\infty(\phi), B_\infty(\phi)$ whenever the dependency in ϕ is relevant.

Proposition 3.4. *The map $h_\gamma^{\mathcal{E}}$ extends to a function on $\mathbb{R}\smallsetminus\{\gamma^{-1}\infty\}\times\{\mathfrak{Re}(s)>0\}$ with the following properties:*

- *for each $x\in\mathbb{R}\smallsetminus\{\gamma^{-1}\infty\}$, the map $s\mapsto h_\gamma^{\mathcal{E}}(x,s)$ is analytic in $\{\mathfrak{Re}(s)>0\}$;*
- *for $\mathfrak{Re}(s)>0$, the map $x\mapsto h_\gamma^{\mathcal{E}}(x,s)$ is continuous on $\mathbb{R}\smallsetminus\{\gamma^{-1}\infty\}$;*
- *given $\varepsilon>0$, uniformly for $\varepsilon\leq\mathfrak{Re}(s)\ll 1$ and $|x-\gamma^{-1}\infty|\geq\varepsilon$, we have*

$$h_\gamma^{\mathcal{E}}(x,s) \ll_\gamma (1+|s|)(1+\left|x-\gamma^{-1}\infty\right|^{\max(\mathfrak{Re}(s)-\frac{1}{2}+\rho_\phi+\varepsilon, 2\mathfrak{Re}(s)-1)}). \tag{3.12}$$

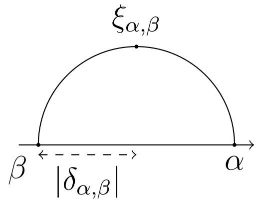
If $s=1/2$, we have the more precise estimate

$$\begin{aligned} h_\gamma^{\mathcal{E}}(x,\tfrac{1}{2}) &= \chi(\gamma)\operatorname{sgn}(c_\gamma)^k\big(\mathcal{M}_{x-\gamma^{-1}\infty}(\phi,\tfrac{1}{2}) - i^k\mathcal{E}_{\gamma^{-1}\infty}(\phi,\tfrac{1}{2})\big)\\ &\quad + O_{\phi,\gamma,\varepsilon}(\left|x-\gamma^{-1}\infty\right|^{-1}).\end{aligned} \tag{3.13}$$

The case $s=1/2$ of this proposition corresponds to [DN26, Theorem 4.1]. Compared with [DN26], the need for uniformity in s requires a delicate analysis of certain terms (denoted $\mathcal{R}_{\alpha,\upsilon}(\phi,s)$ below) which were coincidentally vanishing in the context of [DN26], but do not vanish anymore in our context.

For all $\alpha,\beta\in C(\Gamma)\smallsetminus\{\infty\}$ with $\alpha\neq\beta$, let

$$\delta_{\alpha,\beta} := \tfrac{1}{2}(\alpha-\beta),$$
$$\xi_{\alpha,\beta} := \beta+\delta_{\alpha,\beta}+i|\delta_{\alpha,\beta}|.$$



Hence $\xi_{\alpha,\beta}$ is the point of highest ordinate on the geodesic from α to β.

For $\mathfrak{Re}(s)>1$, let

$$\mathcal{E}^*_{\alpha,\beta}(\phi,s) := \int_\alpha^{\xi_{\alpha,\beta}} \Big(\frac{z-\beta}{|z-\beta|}\Big)^k (\phi(z)-\phi_\infty(\mathfrak{Im}(z)))(\mathfrak{Im}\, z)^{s-1/2}\,\mathrm{d}s(z),$$

where the integral is taken along the geodesic from α to $\xi_{\alpha,\beta}$, which is a quarter-circle, and $\mathrm{d}s(z)$ denotes the $\mathrm{SL}(2,\mathbb{R})$-invariant metric $\mathrm{d}s(z)=\sqrt{\mathrm{d}x^2+\mathrm{d}y^2}/y$. Let also

$$\mathcal{E}_{\alpha,\beta}(\phi,s) := \mathcal{E}^*_{\alpha,\beta}(\phi,s) - (-i\,\mathrm{sgn}(\delta_{\alpha,\beta}))^k\mathcal{E}^*_{\beta,\alpha}(\phi,s). \tag{3.14}$$

Also, given $\upsilon\in\mathrm{SL}(2,\mathbb{R})$, we let $\phi^\upsilon(z)=j_\upsilon(z)^{-k}\phi(\upsilon z)$. Thus ϕ is an automorphic form of same weight, character and eigenvalue as ϕ for the group $\Gamma^\upsilon := \upsilon^{-1}\Gamma\upsilon$, with $\rho_{\phi^\upsilon}=\rho_\phi$. Also note that $C(\Gamma^\upsilon)=\upsilon^{-1}C(\Gamma)$. Finally note that if $\upsilon\in\Gamma$, then $\phi^\upsilon=\chi(\upsilon)\phi$.

Lemma 3.5. *Let $\upsilon\in\mathrm{PSL}(2,\mathbb{R})\smallsetminus\{\left(\begin{smallmatrix}1&\mathbb{R}\\&1\end{smallmatrix}\right)\}$. Denote by $c=c_\upsilon$ the bottom-left coefficient of υ and let $\alpha\in\mathbb{R}\smallsetminus\{\upsilon^{-1}\infty\}$. Assume that $\upsilon\alpha,\upsilon\infty\in C(\Gamma)$. Then for $\mathfrak{Re}(s)>1$ we have*

$$|j(\upsilon,\alpha)|^{2s-1}\mathcal{E}_{\upsilon\alpha}(\phi,s)=\mathrm{sgn}(c)^{-k}\Big(\mathcal{E}_{\alpha,\upsilon^{-1}\infty}(\phi^\upsilon,s)+\mathcal{M}_{\alpha-\upsilon^{-1}\infty}(\phi^\upsilon,s)\Big)+\mathcal{R}_{\alpha,\upsilon}(\phi,s),$$

with the definition (3.11)*, where $\mathcal{R}_{\alpha,\upsilon}(\phi,s)$ is holomorphic in s on $\{\mathfrak{Re}(s)>0\}\cap\mathcal{D}_\phi$, is continuous with respect to α on $\mathbb{R}\smallsetminus\{\upsilon^{-1}\infty\}$ uniformly in compact sets of $\{\mathfrak{Re}(s)>0\}\cap\mathcal{D}_\phi$, and satisfies for $\varepsilon\le\mathfrak{Re}(s)\ll 1$ and $|s-(1\pm it_\phi)|\ge\varepsilon$ the estimate*

$$\begin{aligned}\mathcal{R}_{\alpha,\upsilon}(\phi,s)\ll_{\phi,\varepsilon}(1+|s|)\Big(&|c|^{\frac12-\mathfrak{Re}(s)+\rho_\phi+\varepsilon}|j(\upsilon,\alpha)|^{\mathfrak{Re}(s)-\frac12+\rho_\phi+\varepsilon}+|j(\upsilon,\alpha)|^{2\mathfrak{Re}(s)-1}\\ &+|c|^{1-2\mathfrak{Re}(s)}+|\log(|cj(\upsilon,\alpha)|)||j(\upsilon,\alpha)|^{\mathfrak{Re}(s)-1}|c|^{-\mathfrak{Re}(s)}\Big)\end{aligned}\tag{3.15}$$

uniformly in υ,α and s. Finally, we have $\mathcal{R}_{\alpha,\upsilon}(\phi,\frac12)=0$.

Proof. Denote $\kappa_\upsilon:=\upsilon^{-1}\infty$. For $\mathfrak{Re}(s)>1$ we write

$$\begin{aligned}|j(\upsilon,\alpha)|^{2s-1}\mathcal{E}_{\upsilon\alpha}(\phi,s)=&\int_{\upsilon\alpha}^\infty(\phi(z)-\phi_\infty(\mathfrak{Im}\,z))\Big(|j(\upsilon,\alpha)|^{2s-1}(\mathfrak{Im}\,z)^{s-\frac12}-(\mathfrak{Im}\,\upsilon^{-1}z)^{s-\frac12}\Big)\,\mathrm{d}s(z)\\ &+\int_{\upsilon\alpha}^\infty\phi(z)(\mathfrak{Im}\,\upsilon^{-1}z)^{s-\frac12}\,\mathrm{d}s(z)-\int_{\upsilon\alpha}^\infty\phi_\infty(\mathfrak{Im}\,z)(\mathfrak{Im}\,\upsilon^{-1}z)^{s-\frac12}\,\mathrm{d}s(z)\\ =&\,\mathcal{I}_1+\mathcal{I}_2-\mathcal{I}_3,\end{aligned}$$

say.

We first focus on $\mathcal{I}_2$. Changing variables, we have

$$\begin{aligned}\mathcal{I}_2&=\int_\alpha^{\kappa_\upsilon}j_\upsilon(z)^k\phi^\upsilon(z)(\mathfrak{Im}\,z)^{s-\frac12}\,\mathrm{d}s(z)\\ &=\int_\alpha^{\kappa_\upsilon}j_\upsilon(z)^k(\phi^\upsilon(z)-\phi^\upsilon_\infty(\mathfrak{Im}\,z))(\mathfrak{Im}\,z)^{s-\frac12}\,\mathrm{d}s(z)+\int_\alpha^{\kappa_\upsilon}j_\upsilon(z)^k\phi^\upsilon_\infty(\mathfrak{Im}\,z)(\mathfrak{Im}\,z)^{s-\frac12}\,\mathrm{d}s(z).\end{aligned}$$

The second integral is evaluated as in [DN26, eq. (4.17)]; it equals $\mathrm{sgn}(c)^k\mathcal{M}_{\alpha-\kappa_\upsilon}(\phi^\upsilon,s)$. In the first integral, we split the path of integration as $\alpha\to\xi_{\alpha,\kappa_\upsilon}$ concatenated with $\xi_{\alpha,\kappa_\upsilon}\to\kappa_\upsilon$. Since $j_\upsilon(z)=\mathrm{sgn}(c)\frac{z-\kappa_\upsilon}{|z-\kappa_\upsilon|}$, the first portion contributes $\mathrm{sgn}(c)^{-k}\mathcal{E}^*_{\alpha,\kappa_\upsilon}(\phi^\upsilon,s)$. Finally, for z in the geodesic $\alpha \to \kappa_\upsilon$, we have $\frac{z-\alpha}{|z-\alpha|} = \operatorname{sgn}(\alpha - \kappa_\upsilon) i \frac{z-\kappa_\upsilon}{|z-\kappa_\upsilon|}$, so that the second portion contributes $-(i \operatorname{sgn}(\kappa_\upsilon - \alpha))^k (\operatorname{sgn} c)^{-k} \mathcal{E}^*_{\kappa_\upsilon,\alpha}(\phi^\upsilon, s)$. We thus find

$$\mathcal{I}_2 = \operatorname{sgn}(c)^{-k} \big(\mathcal{E}_{\alpha,\kappa_\upsilon}(\phi^\upsilon, s) + \mathcal{M}_{\alpha-\kappa_\upsilon}(\phi^\upsilon, s) \big),$$

which is our claimed main term.

Let $\mathcal{R}_{\alpha,\upsilon}(\phi, s) := \mathcal{I}_1 - \mathcal{I}_3$. To finish the proof of Lemma 3.5, it remains to show that $\mathcal{I}_1$ and $\mathcal{I}_3$ satisfy the properties claimed about $\mathcal{R}$ in the statement, separately for $\mathcal{I}_3$ and $\mathcal{I}_1$. Consider first $\mathcal{I}_3$, which appears only when ϕ is non-cuspidal. Recall that in that case, we assumed $\mathfrak{Re}(s_\phi) = \frac{1}{2}$. Then, following the estimation of [DN26, eq. (4.7)], we have for $\mathfrak{Re}(s) > 1$ the expression

$$\begin{aligned}\mathcal{I}_3 &= |\alpha - \kappa_\upsilon|^{s-\frac12} \int_0^\infty \phi_\infty\Big(\frac{y}{c^2|\alpha-\kappa_\upsilon|}\Big)\Big(\frac{y}{1+y^2}\Big)^{s-\frac12} \frac{\mathrm{d}y}{y} \\ &= |\alpha - \kappa_\upsilon|^{s-\frac12} \begin{cases} \frac{A_\infty}{(c^2|\alpha-\kappa_\upsilon|)^{s_\phi}} g_{s_\phi}(s) + \frac{B_\infty}{(c^2|\alpha-\kappa_\upsilon|)^{1-s_\phi}} g_{1-s_\phi}(s), & (s_\phi \neq \frac12), \\ \frac{A_\infty - B_\infty \log(c^2|\alpha-\kappa_\upsilon|)}{(c^2|\alpha-\kappa_\upsilon|)^{1/2}} g_{\frac12}(s) + \frac{B_\infty}{(c^2|\alpha-\kappa_\upsilon|)^{\frac12}} g^*_{\frac12}(s), & (s_\phi = \frac12), \end{cases}\end{aligned}$$

with

$$g_\tau(s) := \int_0^\infty \frac{y^{\tau+s-1/2}}{(1+y^2)^{s-1/2}} \frac{\mathrm{d}y}{y} = \frac{\Gamma(\frac{s+\tau-1/2}{2})\Gamma(\frac{s-\tau-1/2}{2})}{2\Gamma(s-1/2)}, \qquad g^*_\tau(s) = \frac{\mathrm{d}}{\mathrm{d}\tau} g_\tau(s).$$

Since $\mathfrak{Re}(s_\phi) = \frac12$, we see from this explicit expression that $\mathcal{I}_3 = \mathcal{I}_3(s,\alpha)$ is holomorphic in s for $\mathfrak{Re}(s) > 0$ and $s \in \mathcal{D}_\phi$. For each such s, it is continuous with respect to α in $\mathbb{R} \smallsetminus \{\kappa_\upsilon\}$. By the Stirling formula, it satisfies for $|s - (1 \pm it_\phi)| \gg 1$ the estimate

$$\mathcal{I}_3 \ll |c|^{-\mathfrak{Re}(s)} |j(\upsilon,\alpha)|^{\mathfrak{Re}(s)-1} (1 + |\log|cj(\upsilon,\alpha))||)(1+|s|)^{-1/2}. \tag{3.16}$$

We clearly have $\mathcal{I}_3(\frac12, \alpha) = 0$, since $g_\tau(\frac12) = g^*_\tau(\frac12) = 0$.

Finally we examine $\mathcal{I}_1 = \mathcal{I}_1(s,\alpha)$, which clearly vanishes at $s = \frac12$. We have

$$|j(\upsilon,\alpha)|^{2s-1} (\mathfrak{Im}\, z)^{s-\frac12} - (\mathfrak{Im}\, \upsilon^{-1} z)^{s-\frac12} = |j(\upsilon,\alpha)|^{2s-1} (\mathfrak{Im}\, z)^{s-\frac12} \Bigg(1 - \bigg|\frac{j(\upsilon^{-1}, z)}{j(\upsilon^{-1}, \upsilon\alpha)}\bigg|^{1-2s}\Bigg),$$

and for $z = \upsilon\alpha + iy$, $y > 0$,

$$\frac{j(\upsilon^{-1}, z)}{j(\upsilon^{-1}, \upsilon\alpha)} = 1 - \frac{icy}{j(\upsilon^{-1}, \upsilon\alpha)} = 1 - icyj(\upsilon,\alpha).$$

Thus, $\Big|\frac{j(\upsilon^{-1},z)}{j(\upsilon^{-1},\upsilon\alpha)}\Big|^2 = 1 + c^2y^2 j(\upsilon,\alpha)^2$ so that

$$\begin{aligned}\Delta(y,s) &:= |j(\upsilon,\alpha)|^{2s-1} (\mathfrak{Im}\, z)^{s-\frac12} - (\mathfrak{Im}\, \upsilon^{-1} z)^{s-\frac12} \\ &= |j(\upsilon,\alpha)|^{2s-1} y^{s-\frac12} \big(1 - (1 + c^2y^2 j(\upsilon,\alpha)^2)^{\frac12 - s}\big) \\ &\ll \big|j(\upsilon,\alpha)^2 y\big|^{\mathfrak{Re}(s)-\frac12} \begin{cases} (1+|s|) c^2 j(\upsilon,\alpha)^2 y^2, & (y < 1/|cj(\upsilon,\alpha)|), \\ 1 + |cj(\upsilon,\alpha)y|^{1-2\,\mathfrak{Re}(s)}, & (\text{otherwise}). \end{cases}\end{aligned}$$

Inserting this estimate in $\mathcal{I}_1$ and using the bound (3.3), we deduce that $\mathcal{I}_1 = \mathcal{I}_1(s,\alpha)$ is holomorphic in $\{\sigma > -\frac32 + \rho_\phi\}$. By uniform convergence, it is a continuous function of α in $\mathbb{R} \smallsetminus \{\kappa_\upsilon\}$, and we have

$$\begin{aligned}\mathcal{I}_1 &\ll \int_0^1 y^{-\rho_\phi+\varepsilon} |\Delta(y,s)| \frac{\mathrm{d}y}{y} + \int_1^\infty \mathrm{e}^{-y/3} |\Delta(y,s)| \frac{\mathrm{d}y}{y} \\ &\ll (1+|s|)|c|^{\frac12 - \mathfrak{Re}(s) + \rho_\phi + \varepsilon} |j(\upsilon,\alpha)|^{\mathfrak{Re}(s) - \frac12 + \rho_\phi + \varepsilon} + |c|^{1-2\,\mathfrak{Re}(s)} + |j(\upsilon,\alpha)|^{2\,\mathfrak{Re}(s)-1}.\end{aligned}$$

Together with (3.16), this provides the claimed bound on $\mathcal{R}_{\alpha,v}(\phi,s)$. $\square$

Lemma 3.6. *Let $\alpha,\beta \in C(\Gamma)\smallsetminus\{\infty\}$ with $\alpha\neq\beta$. Then for $\mathfrak{Re}(s)>1$ we have*

$$\mathcal{E}_{\alpha,\beta}(\phi,s)=\operatorname{sgn}(\delta_{\alpha,\beta})^k\mathcal{E}_{\alpha}(\phi,s)-i^k\mathcal{E}_{\beta}(\phi,s)+\mathcal{T}_{\alpha,\beta}(\phi,s),$$

where $\mathcal{T}_{\alpha,\beta}(\phi,s)$ is holomorphic in s on $\{\mathfrak{Re}(s)>-\frac12+\rho_\phi\}$. For each such s, it is continuous in $\alpha,\beta\in\mathbb{R}$ with $\alpha\neq\beta$ uniformly in s in compact sets, and for $\mathfrak{Re}(s)>-\frac12+\rho_\phi+\varepsilon$, it satisfies

$$\mathcal{T}_{\alpha,\beta}(\phi,s)\ll\begin{cases}|\alpha-\beta|^{-1}, & (|\alpha-\beta|\gg1),\\ |\alpha-\beta|^{\mathfrak{Re}(s)-\rho_\phi-\frac12-\varepsilon}, & (|\alpha-\beta|\ll1).\end{cases}\tag{3.17}$$

Proof. From the definition (3.14) of $\mathcal{E}_{\alpha,\beta}(\phi,s)$, it suffices to prove that the difference

$$\mathcal{E}^*_{\alpha,\beta}(\phi,s)-\operatorname{sgn}(\delta_{\alpha,\beta})^k\mathcal{E}_{\alpha}(\phi,s)\tag{3.18}$$

satisfies the properties claimed on $\mathcal{T}_{\alpha,\beta}(\phi,s)$ in the lemma, and the same for the difference

$$\mathcal{E}^*_{\beta,\alpha}(\phi,s)-\operatorname{sgn}(\delta_{\beta,\alpha})^k\mathcal{E}_{\beta}(\phi,s).$$

By symmetry we may focus *e.g.* on (3.18). We parametrize the arc $\alpha\to\xi_{\alpha,\beta}$ as

$$z_{\alpha,\beta}(y):=\alpha-\nu_{\alpha,\beta}(y)+iy$$

for $y\in(0,\delta_{\alpha,\beta})$, where

$$\nu_{\alpha,\beta}(y)=\delta_{\alpha,\beta}\big(1-(1-y^2/\delta^2_{\alpha,\beta})^{1/2}\big),$$

see [DN26, eq. (3.5)]. The arc length element is $\mu_{\alpha,\beta}(y)\,\mathrm{d}y/y$ with

$$\mu_{\alpha,\beta}(y)=(1-y^2/\delta^2_{\alpha,\beta})^{-1/2}.$$

For clarity we drop the indices α,β in the foregoing computation. A quick computation shows that

$$\nu(y)\ll y^2/|\delta|,\quad \mu(y)-1\ll\frac{y^2}{|\delta|^{3/2}\sqrt{|\delta|-y}},\quad \Big(\frac{z(y)-\beta}{|z(y)-\beta|}\Big)^k-\operatorname{sgn}(\delta)^k\ll y/|\delta|.$$

We find

$$\begin{aligned}&\mathcal{E}^*_{\alpha,\beta}(\phi,s)-\operatorname{sgn}(\delta)^k\mathcal{E}_{\alpha}(\phi,s)\\&\quad=\int_0^{|\delta|}\bigg(\Big(\frac{z(y)-\beta}{|z(y)-\beta|}\Big)^k\mu(y)(\phi(z(y))-\phi_\infty(y))-\operatorname{sgn}(\delta)^k(\phi(\alpha+iy)-\phi_\infty(y))\bigg)y^{s-\frac12}\frac{\mathrm{d}y}{y}\\&\qquad-\operatorname{sgn}(\delta)^k\int_{|\delta|}^\infty(\phi(\alpha+iy)-\phi_\infty(y))y^{s-\frac12}\frac{\mathrm{d}y}{y}.\end{aligned}$$

By (3.3) the second integral is clearly $O(|\delta|^{\mathfrak{Re}(s)-\rho_\phi-\frac12-\varepsilon})$ if $|\delta|<1$ and is $O(|\delta|^{-1})$ otherwise, whereas if $\sigma>\frac12-\rho_\phi$ using the estimates (3.3)-(3.4) we find

$$\begin{aligned}&\int_0^{|\delta|}\bigg\{|\mu(y)-1|y^{-\rho_\phi}\mathrm{e}^{-y/3}+\bigg|\Big(\frac{z(y)-\beta}{|z(y)-\beta|}\Big)^k-\operatorname{sgn}(\delta)^k\bigg|y^{-\rho_\phi}\mathrm{e}^{-y/3}\\&\qquad\qquad+|\phi(z(y)+iy)-\phi(\alpha+iy)|\bigg\}y^{\sigma-1/2+\varepsilon}\frac{\mathrm{d}y}{y}\\&\qquad\ll\int_0^{|\delta|}\Big\{\frac{y^{1-\rho_\phi}\mathrm{e}^{-y/3}}{|\delta|^{3/2}\sqrt{|\delta|-y}}+\frac{y^{1-\rho_\phi}(1+y)\mathrm{e}^{-y/3}}{\delta}\Big\}y^{\sigma-1/2+\varepsilon}\frac{\mathrm{d}y}{y}\\&\qquad\ll|\delta|^{-1}(1+|\delta|^{-1})^{-\sigma-\frac12-\rho_\phi+\varepsilon}.\end{aligned}$$

Since $|\delta| \asymp |\alpha-\beta|$, this implies (3.17). Finally, the continuity in α and β follows by the uniform convergence of the various integrals. □

Proof of Proposition 3.4. Assume $\mathfrak{Re}(s) > 1$. Let $c = c_\gamma$ be the lower-left entry of γ, and note that

$$u_\gamma(x) = (\operatorname{sgn} c)^{-k} \operatorname{sgn}(\delta_{x,\gamma^{-1}\infty})^k \chi(\gamma), \qquad \text{and} \qquad \phi^\gamma = \chi(\gamma)\phi.$$

Using Lemmas 3.5 and 3.6, for $\mathfrak{Re}(s) > 1$ we have

$$\begin{aligned} |j(\gamma,x)|^{2s-1}\mathcal{E}_{\gamma x}(\phi,s) =\ & \operatorname{sgn}(c)^{-k}\chi(\gamma)\big(\mathcal{E}_{x,\gamma^{-1}\infty}(\phi,s) + \mathcal{M}_{x-\gamma^{-1}\infty}(\phi,s)\big) + \mathcal{R}_{x,\gamma}(\phi,s) \\ =\ & u_\gamma(x)\mathcal{E}_x(\phi,s) - i^k(\operatorname{sgn} c)^{-k}\chi(\gamma)\mathcal{E}_{\gamma^{-1}\infty}(\phi,s) \\ & + (\operatorname{sgn} c)^{-k}\chi(\gamma)\big(\mathcal{M}_{x-\gamma^{-1}\infty}(\phi,s) + \mathcal{T}_{x,\gamma^{-1}\infty}(\phi,s)\big) + \mathcal{R}_{x,\gamma}(\phi,s). \end{aligned}$$

This identity holds for $\mathfrak{Re}(s) > 0$, $s \in \mathcal{D}_\phi$ by the same lemmas. Thus, for s outside of an ε-neighborhood of $1 \pm it_\phi$, the properties claimed on the holomorphicity in s and continuity in x follow from the corresponding properties of $\mathcal{R}$ and $\mathcal{T}$ established in Lemmas 3.5 and 3.6, as well as the uniform bound (3.12) for $\mathfrak{Re}(s) \geq \varepsilon$.

Next, using the values given in [DN26, Proof of Proposition 2.3], we compute the polar part of the Laurent expansion of the map $s \mapsto h^{\mathcal{E}}_\gamma(x,s)$ at $s = 1 \pm it_\phi$ and see that these terms vanish. It follows that $h^{\mathcal{E}}_\gamma(x,s)$ is in fact holomorphic at these points.

By the Cauchy formula and the bound (3.12), we deduce the continuity of $x \mapsto h^{\mathcal{E}}_\gamma(x,s)$ at $s = 1 \pm it_\phi$, which finishes the proof of the first two points of Proposition 3.4.

If $s = 1/2$, then by Lemma 3.5 we have $\mathcal{R}_*(\phi, 1/2) = 0$, and so the error term arises from the upper-bound on $\mathcal{T}$ given in Lemma 3.6.

□

3.4. **The period function $h^{\mathcal{E}}$ at parabolic elements.**

Proposition 3.7. *Let $p \in \Gamma$ be a parabolic element with fixed point $\omega \in C(\Gamma) \smallsetminus \{\infty\}$, $\sigma \in \mathrm{SL}(2,\mathbb{R})$ a scaling matrix for ω and $m_0 \in \mathbb{Z}_{\neq 0}$ be such that $\sigma^{-1}p\sigma = \left(\begin{smallmatrix} 1 & m_0 \\ & 1 \end{smallmatrix}\right)$. Then for any $\varepsilon > 0$, there exists $n_0 \geq 1$ such that for $n \geq n_0$ and $|x-\omega| \geq \varepsilon$, the following estimates hold.*

— *For $\varepsilon < \mathfrak{Re}(s) \ll 1$, we have*

$$\begin{aligned} h^{\mathcal{E}}_{p^n}(x,s) \ll_p (1+|s|)\Big(&(1+|x|)^{2\,\mathfrak{Re}(s)-1}\big\{1 + n^{2\,\mathfrak{Re}(s)-1} + n^{\mathfrak{Re}(s)-\frac12+\rho_\phi+\varepsilon}\big\} \\ & + 1 + (1+|x|)^{\mathfrak{Re}(s)-\frac12+\rho_\phi+\varepsilon}\Big). \end{aligned} \tag{3.19}$$

— *For $s = 1/2$, we have*

$$\begin{aligned} h^{\mathcal{E}}_{p^n}(x,\tfrac12) =\ & -(i\operatorname{sgn}(c_\sigma))^k\mathcal{E}_{\sigma^{-1}\infty}(\phi^\sigma,\tfrac12) - u_{p^n}(x)(i\operatorname{sgn}(x-\omega))^k\mathcal{E}_\omega(\phi,\tfrac12) \\ & + \operatorname{sgn}(c_\sigma)^k\mathcal{M}_{nm_0}(\phi^\sigma,\tfrac12) + u_{p^n}(x)\operatorname{sgn}(x-\omega)^k\mathcal{M}_{x-\omega}(\phi,\tfrac12) \\ & + O_p\big(|n|^{-1} + (1+|x|)^{-1}\big). \end{aligned} \tag{3.20}$$

Proof. Assume first $\mathfrak{Re}(s) > 1$, and let $\gamma = p^n$, which is also parabolic with fixed point ω. By picking $n \geq n_0$ large enough in terms of ε, we may suppose that $\gamma^{-1}\infty$ is close enough to ω so as to ensure $|x - \gamma^{-1}\infty| \gg 1$ uniformly in x. We recall that the definition of σ amounts to the conditions $\sigma\infty = \omega$ and $\Gamma_\omega = \sigma\Gamma_\infty\sigma^{-1}$. We also note that $|c_\sigma|$ depends only on ω and thus only on p.

First we use Lemma 3.5 with $\upsilon \leftarrow \sigma$ and $\alpha \leftarrow \sigma^{-1}\gamma x$, getting

$$\begin{aligned} &\left|j(\sigma, \sigma^{-1}\gamma x)\right|^{2s-1}\mathcal{E}_{\gamma x}(\phi, s) \\ &= \operatorname{sgn}(c_\sigma)^{-k}\big(\mathcal{E}_{\sigma^{-1}\gamma x, \sigma^{-1}\infty}(\phi^\sigma, s) + \mathcal{M}_{\sigma^{-1}\gamma x - \sigma^{-1}\infty}(\phi^\sigma, s)\big) + \mathcal{R}_{\sigma^{-1}\gamma x, \sigma}(\phi, s). \end{aligned} \tag{3.21}$$

Next we use Lemma 3.6 with $(\alpha, \beta) \leftarrow (\sigma^{-1}\gamma x, \sigma^{-1}\infty)$. A quick computation shows that $\operatorname{sgn}(\delta_{\sigma^{-1}\gamma x, \sigma^{-1}\infty}) = -\operatorname{sgn}(\delta_{\gamma x, \omega})$, thus we find

$$\begin{aligned} &\mathcal{E}_{\sigma^{-1}\gamma x, \sigma^{-1}\infty}(\phi^\sigma, s) \\ &= (-\operatorname{sgn}(\delta_{\gamma x, \omega}))^k\mathcal{E}_{\sigma^{-1}\gamma x}(\phi^\sigma, s) - i^k\mathcal{E}_{\sigma^{-1}\infty}(\phi^\sigma, s) + \mathcal{T}_{\sigma^{-1}\gamma x, \sigma^{-1}\infty}(\phi^\sigma, s). \end{aligned} \tag{3.22}$$

Next, we use Lemma 3.5 again with $\upsilon \leftarrow \sigma^{-1}\gamma$ and $\alpha \leftarrow x$. Note that $[\phi^\sigma]^{\sigma^{-1}\gamma}(z) = \chi(\gamma)\phi(z)$, and $(\sigma^{-1}\gamma)^{-1}\infty = \gamma^{-1}\omega = \omega$. We obtain

$$\begin{aligned} &\left|j(\sigma^{-1}\gamma, x)\right|^{2s-1}\mathcal{E}_{\sigma^{-1}\gamma x}(\phi^\sigma, s) \\ &= \chi(\gamma)\operatorname{sgn}(c_{\sigma^{-1}\gamma})^{-k}\big(\mathcal{E}_{x,\omega}(\phi, s) + \mathcal{M}_{x-\omega}(\phi, s)\big) + \mathcal{R}_{x, \sigma^{-1}\gamma}(\phi^\sigma, s). \end{aligned} \tag{3.23}$$

Finally, by Lemma 3.6 again, with $(\alpha, \beta) \leftarrow (x, \omega)$, we get

$$\mathcal{E}_{x,\omega}(\phi, s) = \operatorname{sgn}(\delta_{x,\omega})^k\mathcal{E}_x(\phi, s) - i^k\mathcal{E}_\omega(\phi, s) + \mathcal{T}_{x,\omega}(\phi, s). \tag{3.24}$$

We insert the equality (3.24) successively in (3.23), (3.22) and (3.21), and multiply throughout by $|j(\sigma^{-1}\gamma, x)|^{2s-1} = |j(\gamma, x)/j(\sigma, \sigma^{-1}\gamma x)|^{2s-1}$. By the identity

$$\operatorname{sgn}(\delta_{y,\tau^{-1}\infty}) = \operatorname{sgn}(c_\tau)j_\tau(y), \qquad (\tau \in \mathrm{SL}(2,\mathbb{R}), y \neq \tau^{-1}\infty),$$

we get

$$\operatorname{sgn}(\delta_{x,\omega})\operatorname{sgn}(c_{\sigma^{-1}\gamma}) = j_{\sigma^{-1}\gamma}(x), \quad \operatorname{sgn}(\delta_{\gamma x,\omega})\operatorname{sgn}(c_\sigma) = -j_\sigma(\sigma^{-1}\gamma x).$$

Thus, the term $\mathcal{E}_x(\phi, s)$ arises with a factor of exactly $\chi(\gamma)j_\gamma(x)^k = u_\gamma(x)$ by the chain rule, and we get

$$\begin{aligned} h^{\mathcal{E}}_{p^n}(x, s) &= |j(\gamma, x)|^{2s-1}\mathcal{E}_{\gamma x}(\phi, s) - u_\gamma(x)\mathcal{E}_x(\phi, s) \\ &= -(i\operatorname{sgn}(c_\sigma))^k\left|j(\sigma^{-1}\gamma, x)\right|^{2s-1}\mathcal{E}_{\sigma^{-1}\infty}(\phi^\sigma, s) - u_\gamma(x)(i\operatorname{sgn}(\delta_{x,\omega}))^k\mathcal{E}_\omega(\phi, s) \\ &\quad + \left|j(\sigma^{-1}\gamma, x)\right|^{2s-1}\operatorname{sgn}(c_\sigma)^k\mathcal{M}_{\sigma^{-1}\gamma\infty - \sigma^{-1}\infty}(\phi^\sigma, s) + u_\gamma(x)\operatorname{sgn}(\delta_{x,\omega})^k\mathcal{M}_{x-\omega}(\phi, s) \\ &\quad + \left|j(\sigma^{-1}\gamma, x)\right|^{2s-1}\mathcal{R}_{\sigma^{-1}\gamma x, \sigma}(\phi, s) + j_\sigma(\sigma^{-1}\gamma x)^k\mathcal{R}_{x, \sigma^{-1}\gamma}(\phi^\sigma, s) \\ &\quad + \operatorname{sgn}(c_\sigma)^k\left|j(\sigma^{-1}\gamma, x)\right|^{2s-1}\mathcal{T}_{\sigma^{-1}\gamma x, \sigma^{-1}\infty}(\phi^\sigma, s) + u_\gamma(x)\operatorname{sgn}(\delta_{x,\omega})^k\mathcal{T}_{x,\omega}(\phi, s). \end{aligned}$$

Let $\tau = \sigma^{-1}p\sigma = \left(\begin{smallmatrix} 1 & m_0 \\ & 1 \end{smallmatrix}\right)$. Then $\gamma = \sigma\tau^n\sigma^{-1}$. Write $m = m_0 n$, so that $\sigma^{-1}\gamma\sigma = \left(\begin{smallmatrix} 1 & m \\ & 1 \end{smallmatrix}\right)$. Using the property $\sigma\infty = \omega$, we deduce

$$\begin{aligned} &j(\sigma^{-1}\gamma, x) = j(\tau^n\sigma^{-1}, x) = j(\sigma^{-1}, x), \\ &\sigma^{-1}\gamma x - \sigma^{-1}\infty = \tau^n\sigma^{-1}x - \sigma^{-1}\infty = m + \sigma^{-1}x - \sigma^{-1}\infty, \end{aligned}$$

and therefore also $\sigma^{-1}\gamma\infty - \sigma^{-1}\infty = m$. Hence the above simplifies to

$$\begin{aligned} h^{\mathcal{E}}_{p^n}(x, s) &= -(i\operatorname{sgn}(c_\sigma))^k\left|j(\sigma^{-1}, x)\right|^{2s-1}\mathcal{E}_{\sigma^{-1}\infty}(\phi^\sigma, s) - u_\gamma(x)(i\operatorname{sgn}(\delta_{x,\omega}))^k\mathcal{E}_\omega(\phi, s) \\ &\quad + \left|j(\sigma^{-1}, x)\right|^{2s-1}\operatorname{sgn}(c_\sigma)^k\mathcal{M}_m(\phi^\sigma, s) + u_\gamma(x)\operatorname{sgn}(\delta_{x,\omega})^k\mathcal{M}_{x-\omega}(\phi, s) \\ &\quad + \left|j(\sigma^{-1}, x)\right|^{2s-1}\mathcal{R}_{m+\sigma^{-1}x, \sigma}(\phi, s) + j_\sigma(\sigma^{-1}x)^k\mathcal{R}_{x, \tau^n\sigma^{-1}}(\phi^\sigma, s) \\ &\quad + \operatorname{sgn}(c_\sigma)^k\left|j(\sigma^{-1}, x)\right|^{2s-1}\mathcal{T}_{m+\sigma^{-1}x, \sigma^{-1}\infty}(\phi^\sigma, s) + u_\gamma(x)\operatorname{sgn}(\delta_{x,\omega})^k\mathcal{T}_{x,\omega}(\phi, s). \end{aligned} \tag{3.25}$$

By Lemmas 3.2, 3.5 and 3.6, we get that this identity holds for

$$(x,s) \in (\mathbb{R} \smallsetminus \{\gamma^{-1}\infty, \omega\}) \times (\{\mathfrak{Re}(s) > 0\} \cap \mathcal{D}_\phi),$$

and that $h^{\mathcal{E}}_{p^n}(x,s)$ is continuous in x for each s, and holomorphic in s for each x.

Assume first $\mathfrak{Re}(s) \geq \varepsilon$ and $|s - 1 \pm it_\phi| \geq \varepsilon$. Since $|x-\omega| \geq \varepsilon$, we have

$$\left|j(\sigma^{-1}, x)\right| \asymp_p |x - \omega| \gg 1, \qquad \sigma^{-1}x = O(1).$$

Thus, since $c_{\tau^n\sigma} = c_\sigma \asymp_p 1$ and recalling $j(\tau^n\sigma^{-1}, x) = j(\sigma^{-1}, x)$, inserting the bounds (3.5), (3.11), (3.15) and (3.17) in the expression (3.25) yields (3.19). This bound persists for $|s - 1 \pm it_\phi| \leq \varepsilon$ by the maximum modulus principle.

In the special case when $s = 1/2$, the $\mathcal{R}_*$ terms vanish, and since $|x - \omega| \gg 1$ and

$$\left|\sigma^{-1}x + m - \sigma^{-1}\infty\right| = \left|\sigma^{-1}\gamma x - \sigma^{-1}\infty\right| \gg_\sigma \left|x - \gamma^{-1}\infty\right| \gg 1,$$

the estimate (3.20) follows from (3.25). □

Remark 3.8. When ϕ is cuspidal, a change of variables yields

$$\mathcal{E}_{\sigma^{-1}\infty}(\phi^\sigma, \tfrac{1}{2}) = -(-i\,\mathrm{sgn}(c_\sigma))^k \mathcal{E}_\omega(\phi, \tfrac{1}{2}). \tag{3.26}$$

This identity holds also when ϕ is an Eisenstein series, but this requires a more careful treatment of regularized integrals. We will not make use of (3.26) in either case here.

3.5. Uniform Hölder estimates on $h^{\mathcal{E}}$.

Lemma 3.9. *Let p, ω be as in Proposition 3.7, and assume $x, x' \in \mathbb{R} \smallsetminus [\omega - \varepsilon, \omega + \varepsilon]$. For all $n \geq 1$, we have*

$$\left|h^{\mathcal{E}}_{p^n}(x, \tfrac{1}{2}) - h^{\mathcal{E}}_{p^n}(x', \tfrac{1}{2})\right| \ll_{\phi,\varepsilon,\omega} (1 + |x| + |x'|)^{2\varepsilon} n^\varepsilon |x - x'|^{1-\rho_\phi-\varepsilon}. \tag{3.27}$$

Proof. To simplify the notation in the following computation, we denote temporarily $h^{\mathcal{E}}_*(x) = h^{\mathcal{E}}_*(x, \frac{1}{2})$. Since $\omega \neq \infty$ and ω is the unique fixed point of p, we have $\omega \neq p^{-j}\infty$ for all $j \geq 0$. Since $|x - \omega| \geq \varepsilon$, there exists $n_0 > 0$ such that $|x - p^{-n}\infty| \gg 1$ uniformly in $n \geq n_0$ and x, and also that $\mathrm{sgn}(p^n x - p^{-j}\infty)$ is independent of $n \geq n_0$, $j \geq 1$, and x. This implies that $u_{p^j}(p^n x)$ is constant on $\mathbb{R} \smallsetminus [\omega - \varepsilon, \omega + \varepsilon]$ for all $j \geq 0$ and $n \geq n_0$.

By the period relation (3.6), we write

$$h^{\mathcal{E}}_{p^n}(x) = u_{p^{n-n_0}}(p^{n_0}x) h^{\mathcal{E}}_{p^{n_0}}(x) + \sum_{\ell=n_0}^{n-1} u_{p^{n-1-\ell}}(p^{\ell+1}x) h^{\mathcal{E}}_p(p^\ell x)$$

and likewise for x'. Taking differences and using the fact that the various u_{p^j} factors have the same value for x and x' by our choice of n_0, we deduce

$$\left|h^{\mathcal{E}}_{p^n}(x) - h^{\mathcal{E}}_{p^n}(x')\right| \ll \left|h^{\mathcal{E}}_{p^{n_0}}(x) - h^{\mathcal{E}}_{p^{n_0}}(x')\right| + \sum_{\ell=n_0}^{n-1} \left|h^{\mathcal{E}}_p(p^\ell x) - h^{\mathcal{E}}_p(p^\ell x')\right|. \tag{3.28}$$

By our choice of n_0, we have $\left|x - p^{-\ell}\infty\right| \gg 1$ uniformly for $\ell \geq n_0$, and likewise for x', therefore

$$\left|p^\ell x - p^\ell x'\right| \ll \ell^{-2}|x - x'|.$$

By Lemma 3.3, we deduce the bounds

$$\left|h^{\mathcal{E}}_{p^n}(x) - h^{\mathcal{E}}_{p^n}(x')\right| \ll (1 + |x| + |x'|)^{2\varepsilon}|x - x'|^{1-\rho_\phi-\varepsilon}, \qquad (n \leq n_0)$$

$$\left|h^{\mathcal{E}}_p(p^\ell x) - h^{\mathcal{E}}_p(p^\ell x')\right| \ll \left|p^\ell x - p^\ell x'\right|^{1-\rho_\phi-\varepsilon} \ll \ell^{-2+2\rho_\phi+2\varepsilon}|x - x'|^{1-\rho_\phi-\varepsilon},$$

uniformly in ℓ, x and x'. The desired bound follows by picking ε small and inserting in (3.28). □

3.6. **Translation into estimates on the period function of $L(\phi, s, x)$.** We now transfer our estimates from $\mathcal{E}$ to the additive twists (2.6). For all $\gamma \in \Gamma$ and $s \in \mathcal{D}_\phi$, we denote

$$h^L_\gamma(x,s) := |j(\gamma,x)|^{2s-1} L(\phi,s,\gamma x) - j_\gamma(x)^k \chi(\gamma) L(\phi,s,x). \tag{3.29}$$

Lemma 3.10. *For each $\gamma \in \Gamma$, the map $(x,s) \mapsto h^L_\gamma(x,s)$ extends to a map on $(\mathbb{R} \smallsetminus \{\gamma^{-1}\infty\}) \times \{\mathfrak{Re}(s) > 0\}$, which is continuous in x for each s, and holomorphic in s for each x.*

Given $\varepsilon > 0$, $\gamma \in \Gamma$, $x \in \mathbb{R}$ with $|x - \gamma^{-1}\infty| \geq \varepsilon$, and s with $\varepsilon \leq \mathfrak{Re}(s) \ll 1$, we have

$$h^L_\gamma(x,s) \ll_\gamma \frac{\mathrm{e}^{\frac{\pi}{2}|\mathfrak{Im}(s)|}}{(1+|s|)^{\sigma+\mathfrak{Re}(s_\phi)-2}} \cdot \big(1 + |x-\gamma^{-1}\infty|^{\max(\sigma-\frac12+\rho_\phi+\varepsilon, 2\sigma-1)}\big). \tag{3.30}$$

The map $x \mapsto h^L_\gamma(x,\frac12)$ satisfies the Hölder estimate (3.7), with h^L in place of $h^{\mathcal{E}}$.

Moreover, letting for all $x \neq 0$,

$$G_k(\tau,x) = |x|^\tau \frac{(\frac{1+i\,\mathrm{sgn}(x)}{\sqrt{2}})^k \Gamma(\tau)}{2^\tau \Gamma(\frac{k}{2}+\tau)} \mathrm{e}^{-\mathrm{sgn}(x)\frac{\pi i}{2}(\tau-k/2)}, \tag{3.31}$$

we have the following estimates for $|x-\gamma^{-1}\infty| \geq \varepsilon$:

— *When ϕ is cuspidal or when $s_\phi \neq 1/2$,*

$$h^L_\gamma(x,\tfrac12) = u_\gamma(x)\big(A_\infty(\phi)G_k(s_\phi,x) + B_\infty(\phi)G_k(1-s_\phi,x)\big) + O_\gamma(1), \tag{3.32}$$

— *When $s_\phi = 1/2$,*

$$h^L_\gamma(x,\tfrac12) = \begin{cases} u_\gamma(x)G_k(\frac12,x)A_\infty(\phi) + O_\gamma(1), & (B_\infty(\phi)=0), \\ u_\gamma(x)G_k(\frac12,x)B_\infty(\phi)\log|x| + O_\gamma(1+|x|^{1/2}), & (B_\infty(\phi)\neq 0). \end{cases} \tag{3.33}$$

Lemma 3.11. *Let $p, \omega, \sigma, m_0, \varepsilon, x$ and s be as in Proposition 3.7. Then we have*

$$\begin{aligned} h^L_{p^n}(x,s) \ll_p \frac{\mathrm{e}^{\frac{\pi}{2}|\mathfrak{Im}(s)|}}{(1+|s|)^{\sigma+\mathfrak{Re}(s_\phi)-2}} \cdot \Big(&(1+|x|)^{2\mathfrak{Re}(s)-1}\big\{1 + n^{2\mathfrak{Re}(s)-1} + n^{\mathfrak{Re}(s)-\frac12+\rho_\phi+\varepsilon}\big\} \\ &+ (1+|x|)^{\mathfrak{Re}(s)-\frac12+\rho_\phi+\varepsilon}\Big). \end{aligned} \tag{3.34}$$

Furthermore, the Hölder estimate (3.27) holds also for h^L.

Moreover, for all $n \in \mathbb{Z}_{\neq 0}$ and all $|x - p^{-n}\infty| \geq \varepsilon$, we have the following estimates:

— *When ϕ is cuspidal or $s_\phi \neq 1/2$, we have*

$$\begin{aligned} h^L_{p^n}(x,\tfrac12) = {} & \mathrm{sgn}(c_\sigma m_0 n)^k (A_\omega(\phi)G_k(s_\phi, m_0 n) + B_\omega(\phi)G_k(1-s_\phi, m_0 n)) \\ & + u_{p^n}(x)\big(A_\infty(\phi)G_k(s_\phi,x) + B_\infty(\phi)G_k(1-s_\phi,x)\big) \\ & + O_p(1). \end{aligned} \tag{3.35}$$

— *When $s_\phi = 1/2$, define for all $\mathfrak{b} \in C(\Gamma)$ and $x \in \mathbb{R}_{\neq 0}$*

$$D_{\mathfrak{b}}(\phi,x) := \begin{cases} A_{\mathfrak{b}}(\phi)G_k(\frac12,x), & (B_{\mathfrak{b}}(\phi)=0), \\ B_{\mathfrak{b}}(\phi)G_k(\frac12,x)\log|x|, & (B_{\mathfrak{b}}(\phi)\neq 0). \end{cases} \tag{3.36}$$

Then

$$\begin{aligned} h^L_{p^n}(x,\tfrac12) = {} & \mathrm{sgn}(c_\sigma m_0 n)^k D_\omega(\phi, m_0 n) + u_{p^n}(x) D_\infty(\phi,x) \\ & + \begin{cases} O_p(1), & (B_\infty(\phi) = B_\omega(\phi) = 0), \\ O_p(|x|^{1/2} + |n|^{1/2}) & (\textit{otherwise}). \end{cases} \end{aligned} \tag{3.37}$$

We note in passing that for all $x \neq 0$,

$$\Big(\frac{G_k(\frac{1}{2},x)}{G_k(\frac{1}{2},-x)}\Big)^2 = -1. \tag{3.38}$$

Proofs of Lemmas 3.10 and 3.11. By [DN26, eqs. (5.11) & (5.16)], we have

$$L(\phi,s,x) = \begin{cases} \Delta_{it_\phi}(\frac{k}{2},s)^{-1}(g_\phi^+(s)\mathcal{E}_x(\phi,s) + g_\psi^+(s)\mathcal{E}_x(\psi,s)) & (s_\phi \not\equiv k/2 \pmod 1), \\ \Xi_f(s)\mathcal{E}_x(\phi,s) & (s_\phi = k/2), \end{cases} \tag{3.39}$$

where ψ is the weight $k+2$ automorphic form obtained from ϕ by the level-raising operator R_k, and $\Xi_f(s)$, $g_\phi^+(s)$, and $g_\psi^+(s)$ are described in the proof of [DN26, Lemma 5.4] (there $\Xi_f(s)$ is denoted $c_f(s)$), see in particular eqs. (5.18) and (5.21) *ibid.* for the values at $s=1/2$ and Proposition 5.1 and eq. (5.6) for the definitions of Δ_β and Ω_β. Recall that either ϕ is cuspidal or $\mathfrak{Re}(s_\phi) = 1/2$, so that by [DFI02, Corollary 4.4], we have $\mathfrak{Re}(s_\phi) < 1$ whenever $s \not\equiv k/2 \pmod 1$[4]. Using [GR07, (9.111)] and Stirling's formula, we deduce that for $\varepsilon < \mathfrak{Re}(s) \ll 1$

$$\Delta_{it_\phi}(\tfrac{k}{2},s)^{-1}g_\phi^+(s) \ll |\Gamma(s+s_\phi-1/2)|^{-1} \ll \mathrm{e}^{\frac{\pi}{2}|\mathfrak{Im}(s)|}(1+|s|)^{1-\mathfrak{Re}(s+s_\phi)}$$

and the same bounds holds for g_ψ^+ in place of g_ϕ^+ and, by [DN26, eq. (5.20)], for $\Xi_f(s)$.

Thus, the first parts of Lemma 3.10 and Lemma 3.11 follow immediately from Proposition 3.4 by linearity.

It remains to prove (3.32) and (3.35). When ϕ is cuspidal, the main terms on the right-hand side vanish, and the result follows directly from (3.20) by linearity. Suppose then that ϕ is non-cuspidal, in particular $\mathfrak{Re}(s_\phi) = 1/2$ by hypothesis, and assume first $s_\phi \neq 1/2$. From the values in [DN26, Lemma 5.4], we get

$$g_\phi^+(\tfrac{1}{2}) = \frac{\pi^{3/2}2^{1-k/2}}{\sin(\pi s_\phi)}\frac{1}{\Gamma(\frac{k}{4}+\frac{s_\phi}{2})\Gamma(\frac{k}{4}+\frac{1-s_\phi}{2})},$$
$$g_\psi^+(\tfrac{1}{2}) = \frac{-\pi^{3/2}2^{-k/2}}{\sin(\pi s_\phi)}\frac{1}{\Gamma(\frac{1}{2}+\frac{k}{4}+\frac{s_\phi}{2})\Gamma(\frac{1}{2}+\frac{k}{4}+\frac{1-s_\phi}{2})}.$$

Furthermore, by [DFI02, eq. (4.29)],

$$A_\infty(\psi) = (\tfrac{k}{2}+s_\phi)A_\infty(\phi), \quad B_\infty(\psi) = (\tfrac{k}{2}+1-s_\phi)B_\infty(\phi).$$

Thus, by (3.13), inserting the definition (3.10) and using the reflection and duplication formula for the Gamma function, we obtain (3.32) with the claimed value of $G_k(\tau,\eta)$.

Assume next that $s_\phi = 1/2$ and $k=0$. Then by similar computations, using the values [DN26, eq. (4.18)], we obtain

$$h_\gamma^L(x,\tfrac{1}{2}) = u_\gamma(x)\tfrac{1-i\eta}{2}\big(\{A_\infty(\phi) - (\log 2 + i\eta\tfrac{\pi}{2})B_\infty(\phi)\}|\widetilde{x}|^{1/2} + B_\infty(\phi)|\widetilde{x}|^{1/2}\log|\widetilde{x}|\big) + O(1),$$

with $\widetilde{x} := x - \gamma^{-1}\infty$. Assume, finally, that $s_\phi = 1/2$ and $k=1$. Then $s_\phi = k/2$, thus from $\Xi_f(\frac{1}{2}) = (2\pi)^{-1/2}$ and $(\log\Upsilon_{1,\eta})'(\frac{1}{2}) = -3\log 2$, we get

$$h_\gamma^L(x,\tfrac{1}{2}) = u_\gamma(x)\sqrt{\pi}\tfrac{1+i\eta}{2}\big(\{A_\infty(\phi) - 3(\log 2)B_\infty(\phi)\}|\widetilde{x}|^{1/2} + B_\infty(\phi)|\widetilde{x}|^{1/2}\log|\widetilde{x}|\big) + O(1)$$

This proves the asymptotic estimate (3.33).

[4]In [DFI02] the statement restricts to Γ being a congruence subgroup, but the implication we need here follows in general by the same argument.

The estimate (3.35) is proved along a similar argument, using the expansion (3.20) instead of (3.13). We check that for $s_\phi \neq 1/2$ and ϕ non-cuspidal, the corresponding contribution of $\mathcal{M}_{m_0 n}(\dots)$ is

$$(\eta \operatorname{sgn} c_\sigma)^k (A_\omega(\phi) G_k(s_\phi, m_0 n) + B_\omega(\phi) G_k(1 - s_\phi, m_0 n)),$$

where now $\eta = \operatorname{sgn}(m_0 n)$, since $\chi(\gamma) = 1$ unless ω is non-singular in which case $A_\omega(\phi) = B_\omega(\phi) = 0$. Thus, (3.35) follows. □

4. Estimation of a Fourier integral

The case of non-cuspidal forms (Theorem 2.5) will require the order 2 estimation of a Fourier integral, which is the content of the following lemma.

Lemma 4.1. *Let $x_0 \in \mathbb{R}$. Let $(f_n)_n$ and $(W_n)_n$ be sequences of real-valued measurable maps on $[0,1]$ indexed by $n \geq 1$, satisfying the following estimates:*

- *We have $W_n(x) \ll n^{-2}$, with a uniform implied constant.*
- *For some constant $\alpha \geq 0$, we have $\int_0^1 W_n(x)\,\mathrm{d}x = \alpha n^{-2} + O(n^{-3})$.*
- *For some constants $\beta_\pm > 0$, we have $\sum_n W_n(x) = \beta_\pm + O(|x - x_0|)$ for $\operatorname{sgn}(x - x_0) = \pm 1$.*

Assume additionally one of the following hypotheses:

(i) There exist real numbers $a, d_\pm \ll 1$ such that for $\operatorname{sgn}(x - x_0) = \pm 1$ one has

$$f_n(x) = a n^{1/2} + d_\pm |x - x_0|^{-1/2} + O(1);$$

(ii) Or there exist a finite set J and real numbers $a_j, d_{j,\pm} \ll 1$, $\theta_j, \theta_j' \ll 1$ and constants $\tau_j \in \mathbb{R}_{\neq 0}$ such that for $\operatorname{sgn}(x - x_0) = \pm 1$ one has

$$f_n(x) = \sum_{j \in J} a_j n^{1/2} \cos(\tau_j \log n + \theta_j) + d_{j,\pm} |x - x_0|^{-1/2} \cos(\tau_j \log|x - x_0| + \theta_j') + O(1);$$

(iii) Or there exist real numbers $a, d_\pm \ll 1$ such that for $\operatorname{sgn}(x - x_0) = \pm 1$ one has

$$f_n(x) = a n^{1/2} \log n + d_\pm |x - x_0|^{-1/2} \log(|x - x_0|^{-1}) + O(n^{1/2} + |x - x_0|^{-1/2}).$$

Then there exists $\mu \in \mathbb{R}$ and $\sigma \geq 0$ such that uniformly for t in a neighborhood of 0, we have

$$\sum_{n \geq 1} \int_0^1 W_n(x)(\mathrm{e}^{itf_n(x)} - 1)\,\mathrm{d}x = it\mu - \frac{\sigma}{2} t^2 \log(1/|t|)^\xi + O(t^2 (\log 1/|t|)^{\xi - \rho}),$$

where, according to each case, we have

	ξ	ρ	σ
(i)	1	1	$2\alpha\lvert a\rvert^2 + 2\beta \sum_{x_0 \pm 0 \in [0,1]} \lvert d_\pm\rvert^2$
(ii)	1	1	$\sum_{j \in J} \left(\alpha \lvert a_j\rvert^2 + \beta \sum_{x_0 \pm 0 \in [0,1]} \lvert d_{j,\pm}\rvert^2\right)$
(iii)	3	$1 - \varepsilon$	$\frac{8}{3}\alpha\lvert a\rvert^2 + \frac{8}{3}\beta \sum_{x_0 \pm 0 \in [0,1]} \lvert d_\pm\rvert^2$

where the sum over $\pm$ selects the sign(s) for which $[0,1]$ contains a right (if $\pm = +$) or left (if $\pm = -$) neighbourhood of x_0. The implied constant depends at most on α, β, τ_j and on the implied constants in the corresponding hypothesis.

Proof. For simplicity of notation we consider the case $x_0 = 0$ only; the general case follows in the same way. Consider first the case (i). By conjugation, we may assume that $t > 0$. The hypotheses imply the convergence of $\mu := \sum_n \int_0^1 W_n(x) f_n(x)$, and we are left to prove

$$\sum_{n\geq 1} \int_0^1 W_n(x)(\mathrm{e}^{itf_n(x)} - 1)\,\mathrm{d}x = it\mu - \sigma\frac{t^2|\log t|}{2} + O(t^2)$$

where $\sigma = 2(\alpha|a|^2 + \beta|d|^2)$. Write $f_n(x) = an^{1/2} + dx^{-1/2} + g_n(x)$, where by hypothesis $g_n(x)$ is uniformly bounded. Let $\delta_j = \delta_j(n,x)$ be defined as

$$\delta_1 = \mathrm{e}^{iatn^{1/2}} - 1, \qquad \delta_2 = \mathrm{e}^{idtx^{-1/2}} - 1, \qquad \delta_3 := \mathrm{e}^{itg_n(x)} - 1.$$

Then we write

$$\mathrm{e}^{itf_n(x)} - 1 = \delta_1\delta_2\delta_3 + \sum_{i<j} \delta_i\delta_j + \sum_i \delta_i.$$

Since $\delta_i \ll 1$ for each i, and moreover $\delta_i\delta_j \ll t^2 n^{1/2} x^{-1/2}$ whenever $i \neq j$ (here we use $a, d \ll 1$), we deduce that

$$\mathrm{e}^{itf_n(x)} - 1 = O(t^2 n^{1/2} x^{-1/2}) + \sum_i \delta_i.$$

Upon integrating over x and summing over n, the first term contributes $O(t^2)$, and it suffices to prove

$$\sum_n \int_0^1 W_n(x)\big(\mathrm{e}^{ictn^{1/2}} - 1 - iatn^{1/2}\big)\,\mathrm{d}x = -\alpha|a|^2 t^2 |\log t| + O(t^2), \tag{4.1}$$

$$\sum_n \int_0^1 W_n(x)\big(\mathrm{e}^{idtx^{-1/2}} - 1 - idtx^{-1/2}\big)\,\mathrm{d}x = -\beta|d|^2 t^2 |\log t| + O(t^2), \tag{4.2}$$

$$\sum_n \int_0^1 W_n(x)\big(\mathrm{e}^{itg_n(x)} - 1 - itg_n(x)\big)\,\mathrm{d}x = O(t^2). \tag{4.3}$$

The estimate (4.3) is clear by Taylor expansion at order 2 of $\mathrm{e}^{itg_n(x)}$. Regarding (4.1)-(4.2), by conjugation and rescaling t we reduce to the case $a = d = 1$ (here again we use the fact that $a, d \ll 1$).

We apply [BD22a, Proposition 2.2] with $\phi(x) = x^{-1/2}$ and $\mathrm{d}\mu(x) = \sum_n \delta_{x=1/n} \int_0^1 W_n(y)\,\mathrm{d}y$, with $\delta_{x=1/n}$ being the Dirac mass at $1/n$, and $\mathrm{d}\mu(x) = \sum_n W_n(x)\,\mathrm{d}x$ respectively, and parameters given by $(\alpha, \xi, \rho) \leftarrow (2, 1, 1)$.

In the case (ii), a similar argument reduces the problem to the analysis of

$$\sum_n \int_0^1 W_n(x)(\mathrm{e}^{itn^{1/2}\cos(\tau\log n + \theta)} - 1 - itn^{1/2}\cos(\dots))\,\mathrm{d}x, \tag{4.4}$$

and corresponding analogues of (4.2) and (4.3). We apply [BD22a, Proposition 2.2] with $\phi(x) = x^{-1/2}\cos(\tau\log(1/x) + \theta)$, the same choices of measures $\mathrm{d}\mu(x)$ and same parameters.

Finally, in the case (iii), the same argument goes through by applying [BD22a, Proposition 2.2] with $\phi(x) = x^{-1/2}\log(1/x)$, with parameters $(\alpha, \xi, \rho) \leftarrow (2, 3, 1 - \varepsilon)$. $\square$

Remark 4.2. Taking $W_n = 0$ for all sufficiently large n, we see that the Lemma holds, with $\alpha := 0$, when $(W_n)_n$, $(f_n)_n$ are finite sequences. Moreover, with obvious modifications, the lemma remains valid when the integration is taken over any regular bounded curve rather than over $[0,1]$. This follows by parametrizing the curve and applying the lemma to the resulting integral.

5. Translation to the disk model

To use the results in [BDLb], it will be more convenient to use the disk model for the hyperbolic plane. Define

$$p_{\mathbb{D}} : \overline{\mathbb{H}} \to \overline{\mathbb{D}}, \qquad p_{\mathbb{D}}(z) = \frac{z-i}{z+i} \tag{5.1}$$

which sends $\partial\mathbb{H}$ to $\partial\mathbb{D}$. Its inverse is given by

$$p_{\mathbb{H}}(z) = i\frac{1+z}{1-z}. \tag{5.2}$$

With a slight abuse of notation, we indicate with $p_{\mathbb{D}}$ and $p_{\mathbb{H}}$ the matrices in $\mathrm{SL}(2,\mathbb{C})$ corresponding to these Möbius transformation. For $z \in \mathbb{C} \cup \{\infty\}$, we also denote

$$[z]_{\mathbb{D}} = p_{\mathbb{D}} z, \quad [z]_{\mathbb{H}} = p_{\mathbb{H}} z,$$

while for $\gamma \in \mathrm{SL}(2,\mathbb{R})$ and $g \in \mathrm{SU}(1,1)$, we denote

$$[\gamma]_{\mathbb{D}} = p_{\mathbb{D}} \gamma p_{\mathbb{H}}, \qquad [g]_{\mathbb{H}} = p_{\mathbb{H}} g p_{\mathbb{D}}.$$

We use the same notation for sets of matrices. Let $\overline{G} := [\Gamma]_{\mathbb{D}} = p_{\mathbb{D}} \Gamma p_{\mathbb{D}}^{-1}$ be the image of Γ in the disk model. The group $\overline{G} \subset \mathrm{SU}(1,1)$ is discrete, finitely generated, and has finite co-volume, since the same is true of Γ. Since Γ has a cusp at ∞ of width 1, we deduce that $\overline{G}$ has a cusp at 1 of width 1.

In order to apply the results of Section 9, it will be convenient to work with a subgroup of sufficiently large rank.

Lemma 5.1. *The group $\overline{G}$ admits a subgroup G of finite index which is free, of rank at least 6, and invariant under conjugation,*

$$\left(\begin{smallmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{smallmatrix}\right) \in G \iff \left(\begin{smallmatrix} \overline{\alpha} & \overline{\beta} \\ \beta & \alpha \end{smallmatrix}\right) \in G. \tag{5.3}$$

Proof. By *e.g.* [Sel60, Lemma 8], $\overline{G}$ admits a torsion-free subgroup G of finite index[5], which is therefore free [New85, p. 683] of some rank n (see [EEK82, p. 335] for a short algebraic proof in our non-cocompact case).

If $n \geq 6$ and G is self-conjugate, then G meets our requirements. Otherwise, if G is freely generated by $g_1, \dots, g_s$, say, we let $G' = \{\prod_i g_i^{n_i}, \sum_i n_i \equiv 0 \pmod 6\}$ and $G'' := G' \cap rG'r$, where $r = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then G'' is a self-conjugate subgroup of G of finite index at least 6 in G. Therefore by [MKS04, Corollary 2.9, Theorem 2.10], it is free of rank at least 6. ∎

In the following we shall fix a G satisfying the properties of Lemma 5.1. Note that the group G is discrete, of covolume $[\overline{G} : G]\,\mathrm{Vol}(\mathbb{H}/\overline{G}) < \infty$ [Kat92, Theorem 3.1.2]. It inherits a cusp at 1 from $\overline{G}$, and thus satisfies all the hypotheses from [BDLb, Section 1.1].

Finally, for $G' \in \{G, \overline{G}\}$, $\mathfrak{b} \in C(G')$ and $Q > 0$, we define the analogue of (2.7) for subgroups of $\mathrm{SU}(1,1)$ as

$$C_{\mathfrak{b}}(G';Q) := \{x \in G'\mathfrak{b}, 0 < |\,\mathfrak{Im}\,\Upsilon_{\mathfrak{b}}^{G'}(x)| \leq Q\}$$

where, following [BDLb, eq. (1.5)], $\Upsilon_{\mathfrak{b}}(x)$ is defined for $x = g'\mathfrak{b}$, with $g' \in G'$ as

$$\Upsilon_{\mathfrak{b}}^{G'}(x) := \overline{\alpha + \beta}, \qquad g' s'_{\mathfrak{b}} = \left(\begin{smallmatrix} \alpha & \beta \\ * & * \end{smallmatrix}\right),$$

[5] The constructions in [Sel60, EEK82] yield a normal subgroup, but we will not use this property

where $s'_{\mathfrak{b}} \in \mathrm{SU}(1,1)$ is a scaling matrix for $\mathfrak{b}$ in G', which in this context means $[s'_{\mathfrak{b}}]_{\mathbb{H}}$ is a scaling matrix for $[\mathfrak{b}]_{\mathbb{H}}$, that is

$$(5.4) \qquad s'_{\mathfrak{b}}1 = \mathfrak{b}, \qquad G'_{\mathfrak{b}} = s'_{\mathfrak{b}} \left\{ \pm \left(\begin{smallmatrix} 1+ki/2 & -ki/2 \\ ki/2 & 1-ki/2 \end{smallmatrix} \right) \middle| \, k \in \mathbb{Z}, \ \pm = + \text{ if } G' = G \right\} s'^{-1}_{\mathfrak{b}}.$$

A quick computation shows $\mathfrak{Im}\, \Upsilon^G_{\mathfrak{b}}(g\mathfrak{b}) = c_{[g]_{\mathbb{H}}[s_{\mathfrak{b}}]_{\mathbb{H}}}$; in particular, for any $\mathfrak{b} \in C(\Gamma)$ one has

$$(5.5) \qquad C_{\mathfrak{b}}(\Gamma; Q) = [C_{[\mathfrak{b}]_{\mathbb{D}}}(G; Q)]_{\mathbb{H}}.$$

Lemma 5.2. *Let $(\overline{g}_j)_j$ be a set of representatives of $G\backslash\overline{G}$ and, given $\mathfrak{b} \in C(G)$. Then:*

— *For all j, we have $\overline{g}_j\mathfrak{b} \in C(G)$.*

— *Let $n_j \in \mathbb{Z}_{>0}$ be the index $|G_{\mathfrak{b}} : \overline{g}_j^{-1} G_{\overline{g}_j\mathfrak{b}}\, \overline{g}_j|$. Then for all $Q > 0$, we have*

$$C_{\mathfrak{b}}(\overline{G}; Q) = \bigcup_j C_{\overline{g}_j\mathfrak{b}}(G; Q\sqrt{n_j})$$

where the union is disjoint.

Proof. Let $\mathfrak{b}_j := \overline{g}_j\mathfrak{b}$. Note that obviously $\mathfrak{b}_j \in C(\overline{G})$. To prove the first point, let $\sigma \in G$ be parabolic and such that $\sigma\mathfrak{b} = \mathfrak{b}$, and $\tau = \overline{g}_j\sigma\overline{g}_j^{-1}$, so that $\tau\mathfrak{b}_j = \mathfrak{b}_j$. Since $\overline{G}/G$ is finite, we may find a positive integer $n > 0$ and $\pi \in G$ such that $\tau^n \in G$. The equality $\tau^n\mathfrak{b}_j = \mathfrak{b}_j$ and the fact that τ^n is a parabolic element of G implies that $\mathfrak{b}_j \in C(G)$. By definition of n_j, we have

$$G_{\mathfrak{b}_j} = \overline{g}_j\overline{s}_{\mathfrak{b}}\{\left(\begin{smallmatrix} 1+ki/2 & -ki/2 \\ * & * \end{smallmatrix} \right),\, k \in n_j\mathbb{Z}\}(\overline{g}_j\overline{s}_{\mathfrak{b}})^{-1},$$

where $\overline{s}_{\mathfrak{b}}$ is a scaling matrix for $\mathfrak{b}$ as a cusp of $\overline{G}$. We conjugate by $t := \left(\begin{smallmatrix} (1+n_j)/2\sqrt{n_j} & (1-n_j)/2\sqrt{n_j} \\ * & * \end{smallmatrix} \right)$, which gives the following scaling matrix for $\mathfrak{b}_j$ as a cusp of G,

$$s_{\mathfrak{b}_j} = \overline{g}_j\overline{s}_{\mathfrak{b}}t.$$

Let now $x = g\mathfrak{b}_j = g\overline{g}_j\mathfrak{b}$ for some $g \in G$ and let $\left(\begin{smallmatrix} \alpha' & \beta' \\ * & * \end{smallmatrix} \right) := g\overline{g}_j\overline{s}_{\mathfrak{b}}$ and $\left(\begin{smallmatrix} \alpha & \beta \\ * & * \end{smallmatrix} \right) = g\overline{g}_j\overline{s}_{\mathfrak{b}}t$. Clearly, $\left(\begin{smallmatrix} \alpha & \beta \\ * & * \end{smallmatrix} \right) = \left(\begin{smallmatrix} \alpha' & \beta' \\ * & * \end{smallmatrix} \right)t$, which implies $\alpha + \beta = (\alpha' + \beta')/\sqrt{n_j}$. Thus,

$$\Upsilon^G_{\mathfrak{b}_j}(x) = \frac{1}{\sqrt{n_j}}\Upsilon^{\overline{G}}_{\mathfrak{b}}(x).$$

Since we have a disjoint union $\overline{G} = \cup_j\overline{g}_jG$, the conclusion of the lemma follows. □

5.1. **The skew product.** We need also to transfer the multiplier $u_\gamma(z)$ defined in (2.1) to the disk setting. For all $g \in G$ and $x \in \mathbb{D} \smallsetminus \{1, g^{-1}1\}$, let

$$\vartheta_g(x) := \chi'(g)j_\gamma(y)^{-k},$$

where $y = [x]_{\mathbb{H}}$, $\gamma = [g]_{\mathbb{H}}$ and $\chi'(g) := \overline{\chi(\gamma)}$. With this correspondence, we have

$$\vartheta_g(x) = \overline{u_\gamma(y)}$$

and the period relation

$$(5.6) \qquad \vartheta_{g_1g_2}(x) = \vartheta_{g_1}(g_2x)\vartheta_{g_2}(x),$$

which we will use repeatedly. Note that the set of values taken by $\vartheta_g(x)$ forms a subgroup of the roots of unity, namely

$$\Theta = \{\vartheta_g(x), g \in G, x \in C(G) \smallsetminus \{1, g^{-1}1\}\} = \{\eta^k\chi(\gamma), (\gamma, \eta) \in \Gamma \times \{\pm 1\}\}.$$

6. Transfer operators

6.1. **The operator $\mathscr{C}$.** We now summarize the set-up of [BDLb], which builds on [BS79, Mar18]. We let R be a fundamental domain for $G\backslash\mathbb{D}$ whose sides are geodesics. By [Tuk73, Proposition] we may pick R having all its vertices on $\mathbb{D}$ and, by rotating the domain if necessary, we can assume 1 is a vertex. Each side s is paired with another side of R, say $\widehat{s}$, by an element $g_s \in G \smallsetminus \{1\}$, in such a way that $s = g_s\widehat{s}$ and $g_{\widehat{s}} = g_s^{-1}$; finally the group G is a free product of the elements g_s.

We label the sides of R according to elements belonging to a finite set Σ of letters – which we identify with $\mathbb{Z}/2m\mathbb{Z}$ for some $m \geq 1$ – proceeding counterclockwise from 1. For convenience of notation we import the additive structure of $\mathbb{Z}/2m\mathbb{Z}$ and in particular, we have that $a, b \in \Sigma$ are contiguous sides if and only if $a - b = \pm 1$.

Given a word $w = a_1 \cdots a_\ell$ with $a_j \in \Sigma$, we say that w is *admissible* if $a_j \neq \widehat{a_{j+1}}$ for all j. We denote by Σ^* the set of admissible (finite) words in Σ. Given $w = a_1 \cdots a_\ell \in \Sigma^*$, we let $g_w := g_{a_1} \cdots g_{a_\ell}$. We also let $\widehat{w} := \widehat{a}_\ell \cdots \widehat{a}_1$, so that $g_{\widehat{w}} = g_w^{-1}$, and we denote by h_w the Möbius transformation associated to g_w, that is $h_w(z) := g_w \cdot z$.

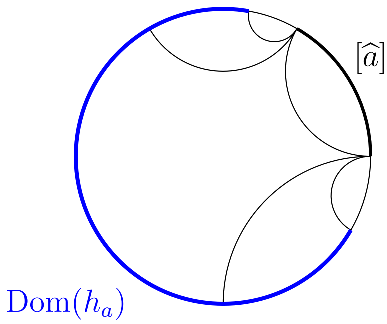
Given $\ell \geq 1$, we say that $w = a_1 \cdots a_\ell \in \Sigma^*$ is a *cuspidal cycle* if for some *direction* $\delta \in \{\pm 1\}$ one has

$$a_{j+1} = \widehat{a}_j + \delta \quad \forall j = 1, \ldots, \ell - 1, \qquad a_1 = \widehat{a}_\ell + \delta, \tag{6.1}$$

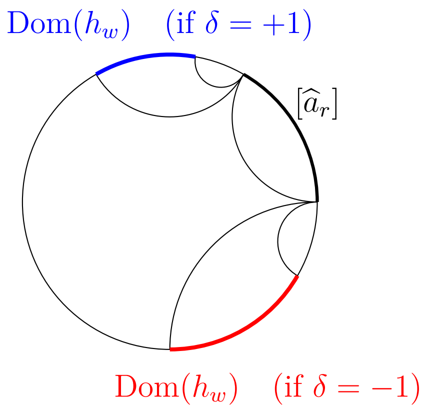
see the corresponding definition in [Mar18, Section 3.2]. We say that w_1 is a non-empty prefix of w if $w_1 = a_1 \cdots a_j$ with $1 \leq j \leq \ell$, and say the cuspidal cycle w is minimal if there is no non-trivial, non-empty prefix of w which is cuspidal. Given $a \in \Sigma$, there are exactly two minimal cuspidal cycles with first letter a, one for each choice of the direction δ.

Given $a \in \Sigma$, let $[a]$ and $[a]^\circ$ be respectively the closed and open arcs in $\partial\mathbb{D}$ delimited by the two endpoints of a and not intersecting any other vertex of R. We have that $g_a(\partial\mathbb{D} \smallsetminus [\widehat{a}]^\circ) = [a]$. More generally, if $w = a_1 \cdots a_\ell \in \Sigma^*$, then $g_w(\partial\mathbb{D} \smallsetminus [\widehat{a}_\ell]^\circ)$ is a sub-arc of $[a_1]$ which we denote with $[a_1, \ldots, a_\ell]$ or $[w]$. As the length of w goes to infinity, the arc $[a_1, \ldots, a_\ell]$ shrinks and determines a unique limiting point indicated by $[a_1, a_2, \ldots]$. Every element of $\partial\mathbb{D}$ can be written in this way, although not necessarily in a unique way (see [BS79]). The Bowen-Series map $\mathcal{B}$ is the map that sends $[a_1, a_2, \ldots]$ to $[a_2, a_3, \ldots]$. This is well-defined on $\partial\mathbb{D}$ except on the set V of vertices of R. Because of the presence of cusps, the map $\mathcal{B}$ is not uniformly expanding. As in [BDLb], we consider two "accelerations" of $\mathcal{B}$: the induced map $\mathcal{F}_K$ introduced in [BS79], and the cuspidal acceleration $\mathcal{F}_c : \partial\mathbb{D} \smallsetminus V \to \partial\mathbb{D}$ defined in [BDLb], which is related to the cusp acceleration introduced in [AMU16]. We refer to [BDLb, Section 3] for the precise definition, and recall first the classification of the inverse branches $\mathcal{H}_c$ of $\mathcal{F}_c$. These are Möbius transformations defined on suitable subsets of $\partial\mathbb{D}$ and can be divided in the following types :

(i) The maps $h = h_w$ for $w = a \in \Sigma$, with domain $\mathrm{Dom}(h)$ consisting of the union of all arcs of the shape $[b_1, b_2]$, $b_1 \neq \widehat{a}$, which are not contiguous to $[\widehat{a}]$. This last condition means that either $b_1 \neq \widehat{a} \pm 1$, or else $b_2 \neq \widehat{b_1 + (b_1 - \widehat{a})}$.



(ii) For each triple (w_0, w_1, n) with w_0 a minimal cuspidal cycle, w_1 a non-empty prefix of w_0, and $n \in \mathbb{Z}_{\geq 0}$ such that $n \geq 1$ if w_1 is of length 1, the map $h = h_w$ where $w = w_0^n w_1$. Its domain $\mathrm{Dom}(h)$ consists of the union of all arcs $[\widehat{a}_r + \delta, b_2]$ which are not contiguous to $\widehat{a}_r$, where a_r is the last letter of w_1 and δ is the direction of w_0. The maps $h = h_w$ for $w = a \in \Sigma$, with domain $\mathrm{Dom}(h)$ consisting of the union of all arcs of the shape $[b_1, b_2]$, $b_1 \neq \widehat{a}$, which are not contiguous to $[\widehat{a}]$. This last condition means that either $b_1 \neq \widehat{a} \pm 1$, or else $b_2 \neq \widehat{b}_1 + (b_1 - \widehat{a})$.



For simplicity of exposition, we shall often indicates also the index of the inverse branches of the first kind as $w = w_0^n w_1$ with $n = 0$, $w_1 = a$ and w_0 a minimal cuspidal cycle such that w_1 is a non-trivial prefix of w_0. By [BDLb, Corollary 4.6], for $h \in \mathcal{H}_c$ and $x \in \mathrm{Dom}(h)$, we have $|h'(x)| \asymp 1/(n+1)^2$ where n is determined above. In particular this is independent of x, and for brevity, we denote

$$|h'| \asymp |h'(x)|, \qquad (x \in \mathrm{Dom}(h))$$

in all estimates where an order of magnitude is sought.

Notice that the set of possible w_0 and w_1 that appear in these decompositions is finite. We also remark that by [BDLb, Lemma 4.4] the matrix g_{w_0} is parabolic and generate the stabilizer of its fixed point $\omega = \omega_{w_0} = [w_0, w_0, w_0, \dots]$. The fixed points ω_{w_0} are vertices of R and, as w_0 varies, they generate $G \backslash C(G)$. For each vertex ω of R, we fix a scaling matrix s_ω and let $\upsilon_{w_0} \in \{\pm 1\}$, $\iota_{w_0} \in \{0, (\pm 1)^k\}$ be defined as

$$[s_{\omega_{w_0}}^{-1} g_{w_0} s_{\omega_{w_0}}]_{\mathbb{H}} = \begin{pmatrix} 1 & \upsilon_{w_0} \\ & 1 \end{pmatrix} \qquad \iota_{w_0} := \begin{cases} \mathrm{sgn}(\upsilon_{w_0} c_{[s_\omega]_{\mathbb{H}}})^k & \omega \neq 1, \\ 0 & \omega = 1. \end{cases} \tag{6.2}$$

We remark that $\omega_{w_0} = \omega_{\widehat{w}_0}$ and $\upsilon_{w_0} = -\upsilon_{\widehat{w}_0}$.

Furthermore, for any $h \in \mathcal{H}_c$, we let $\mathrm{Im}(h) := h(\mathrm{Dom}(h))$. For any $w = a_1 \cdots a_\ell \in \Sigma^*$ we have that $\mathrm{Im}(h_w)$ is contained in the arc $[a_1]^\circ$, and is bounded away from its endpoints by a constant (depending on w). In particular, for any $w = w_0^n w_1$ as above we have

$$|g_w x - 1| \gg_n 1,\ |g_{w_1} x - 1| \gg 1,\ |g_{w_1} x - \omega_{w_0}| \gg 1 \quad \forall x \in \mathrm{Dom}(h_w), \tag{6.3}$$

where the second and third bounds hold since $\mathrm{Dom}(h_w) \subseteq \mathrm{Dom}(h_{w_1})$. This also implies

$$|x - g_w^{-1} 1| \gg_n 1,\ |x - g_{w_1}^{-1} 1| \gg 1,\ |g_w x - g_{w_0}^{-n} 1| \gg 1 \quad \forall x \in \mathrm{Dom}(h_w). \tag{6.4}$$

In addition to the construction above quoted from [BDLb], we will need the further hypothesis that R is conjugation invariant, that is

$$\overline{R} = R. \tag{6.5}$$

To obtain such a domain, it suffices to follow Tukia's construction [Tuk72, Tuk73] of the fundamental domain R maintaining the symmetry at each step, as allowed by the fact that G is conjugation invariant. We remark that the assumption (6.5) is not preserved by rotations of the fundamental domain. Nonetheless, we still have that 1 is a vertex of R. Indeed, if we assume by contradiction there is a (necessarily symmetric) side s encircling 1, then also the side $\widehat{s}$ would have to be symmetric and thus, since $\widehat{s} \neq s$, it would have to encircle -1. Since $\overline{g}_s$ also sends $\widehat{s}$ to s then, by unicity, g_s is real and thus fixes 1; moreover, as the two sides are not tangent, g_s is hyperbolic. Since 1 is a cusp, it is then fixed also by a parabolic matrix, which is impossible.

6.2. **The operator $\mathscr{L}$.** Next we recall from [BDLb] the definitions of the map $\mathcal{F}_K$, the set $\mathcal{H}$ of its inverse branches, and its transfer operator $\mathcal{L}$. In the present paper we will only need a few facts on the sets $\mathcal{H}$ and $\mathscr{L}$.

Let

$$K = \bigcup_{\substack{a,b \in \Sigma \\ |a - \widehat{b}| > 1}} [a, b].$$

The set $\mathcal{H}$ is a set of smooth maps from K to itself, satisfying the following:

- We have $\min_K |h'| \asymp \max_K |h'|$, which justifies the notation $|h'|$ defined up to a constant depending on R.
- For some $\delta > 0$, we have
$$\sum_{h \in \mathcal{H}} |h'|^{1-\delta} < \infty.$$
- Every $h \in \mathcal{H}$ decomposes as compositions of maps from $\mathcal{H}_c$, in the way described in [BDLb, Lemma 3.3].

We can now quote from [BDLb, Section 5.4] the following definition. For all $s \in \mathbb{C}$ with $\mathfrak{Re}(s) > 1 - \delta$, the operator $\mathscr{L}_s$ is defined on maps on $\partial\mathbb{D} \times \Theta$ by

$$\mathscr{L}_s[f](y, \vartheta_1) = \sum_{\substack{h \in \mathcal{H} \\ y \in \mathrm{Dom}(h)}} |h'(y)|^s f(h(y), \vartheta_1 \vartheta_h(y)), \qquad (y \in K, \vartheta \in \Theta)., \tag{6.6}$$

where for $h = h_w \in \mathcal{H}$

$$\vartheta_h(y) := \vartheta_{g_w}(y).$$

For all $(x, \vartheta) \in K \times \Theta$, let

$$\mathcal{F}_K(x, \vartheta) := (\mathcal{F}_K(x), \vartheta\vartheta_g(x)), \qquad (\mathcal{F}_K(x) = gx, g \in G), \tag{6.7}$$

and denote

$$\mathrm{d}\nu(x, \vartheta) = \mathrm{d}s(x)\,\mathrm{d}\vartheta, \tag{6.8}$$

where $\mathrm{d}\vartheta$ is the normalized counting measure.

By definition we have $\mathcal{F}_K(h(x)) = x$, and thus

$$\int_{K\times\Theta} \mathscr{L}_s[f]g \,\mathrm{d}\nu = \int_{K\times\Theta} \mathscr{L}_1[f\cdot|\mathcal{F}_K'|^{1-s}\cdot g\circ\mathcal{F}_K] \,\mathrm{d}\nu = \int_{K\times\Theta} f\cdot(g\circ\mathcal{F}_K)\cdot|\mathcal{F}_K'|^{1-s}\,\mathrm{d}\nu, \tag{6.9}$$

by [BDLb, (5.8)].

The properties of $\mathscr{L}_s$ play a key role in our work. For the most part we will be able to directly import the required information from [BDLb, Theorem 1.3]. However, we shall still need some key facts on $\mathscr{L}_s$ when computing the first and second moments in Section 8.

Lemma 6.1. *There exists $\delta \in (0, \frac{1}{10})$, and a bounded open set $\mathcal{U} \subseteq \mathbb{C}$ contained in $\{\mathfrak{Re}(s) > 1-\delta\}$ and containing the interval $(1-\delta, 2]$, for which the following holds:*

(i) For each $s \in \mathcal{U}$, the operator $\mathscr{L}_s$ acting on

$$C^1(K\times\Theta, \|\cdot\|_1), \qquad \|f\|_1 = \|f\|_\infty + \|f'\|_\infty$$

is precompact, where derivative is understood with respect to the first variable and the norms are taken over $K\times\Theta$. It has a simple eigenvalue $\lambda(s)$ of largest modulus.

(ii) Denote ξ_s the eigenfunction associated to $\lambda(s)$ with normalization $\xi_s(0,\vartheta) = \xi_1(0,\vartheta)$. Then ξ_s depends only on the first variable; we have $\xi_s(x) = O(1)$ uniformly for $x \in K$ and $s \in \mathcal{U}$. The same holds for $\xi_s'(x)$.

(iii) The map $s \mapsto \xi_s(x)$ is holomorphic on $\mathcal{U}$ for all $x \in K$.

Proof. (i) For s in a neighbourhood of 1, this is [BDLb, Proposition 7.3] with the difference that there $\mathcal{L}_s$ acts on a suitable Hölder space. The same proof gives the statement in C^1 (*cf.* also [LS25, Proposition 7.5]). If $s \in \mathbb{R}_{>1}$, then the result follows from the classical Perron-Frobenius theorem, as in [Bal00, Theorem 1.5]. Finally, by perturbation theory [Kat95, Theorem VII.1.7], we obtain the conclusion for $s \in \mathcal{U}$ if $\mathcal{U}$ is a small enough neighborhood of $(1-\delta, 2]$.

(ii) The first point follows since the averaging operator $f(x,\vartheta_1) \mapsto |\Theta|^{-1}\sum_{\vartheta\in\Theta} f(x,\vartheta)$ commutes with $\mathcal{L}_s$ for all s. To see the second point, first note that by [Kat95, Theorem VII.1.7], the essential spectral gap ρ_s is bounded below by a positive constant for $s \in \mathcal{U}$. Let $\mathscr{P}_s$ be the eigenprojection associated to $\lambda(s)$. Since $s \mapsto \mathscr{L}_s$ is bounded-holomorphic in the sense of *ibid.*, we deduce that $s \mapsto \mathscr{P}_s$ also is, reducing the neighborhood $\mathcal{U}$ if needed. The claim then follows since the eigenfunction ξ_s defined in the statement is $\xi_s = c\mathscr{P}_s[\mathbf{1}]$ for some constant $c > 0$, therefore $\|\xi_s\|_1 \ll \|\mathscr{P}_s\| \ll 1$.

(iii) This follows from the bounded-holomorphicity of $s \mapsto \mathscr{P}_s$ in the previous paragraph.

□

7. Proofs of the main statements: limiting theorems

In this section we begin the proof of Theorems 2.2 and 2.5.

7.1. Estimates on the observable.

We translate the estimates of Lemmas 3.10 and 3.11 in terms of the disk model. For $x \in C(G)$ and $g \in G$, recalling the notation (5.2), we let

$$\begin{aligned} \boldsymbol{L}(x,s) &:= |p_{\mathbb{H}}'(x)|^{1-s} L(\phi, s-\tfrac{1}{2}, [x]_{\mathbb{H}}) \in \mathbb{C}, && (x \neq 1) \\ \boldsymbol{\phi}_g(x,s) &:= \vartheta_g(x)\boldsymbol{L}(gx,s) - |g'(x)|^{s-1}\boldsymbol{L}(x,s), && (x \notin \{1, g^{-1}1\}). \end{aligned} \tag{7.1}$$

We warn the reader about the shift in the parameter s, so that the additive twist $L_\phi(x)$ now correspond to $s=1$. This choice is more natural with respect to the normalization in [BV05, BDLb]. We stress the difference in notation between the automorphic form ϕ, and the observable $\boldsymbol{\phi}_g$ (in bold, indexed by elements of G), which will also be clear from the context.

The quantities (7.1) are initally defined for $\mathfrak{Re}(s) > 3/2$, in which domain we have

$$\boldsymbol{\phi}_g(x,s) = |p'_{\mathbb{H}}(x)|^{1-s}|g'(x)|^{s-1}\vartheta_g(x)h^L_\gamma([x]_{\mathbb{H}}, s-\tfrac12), \qquad (\gamma=[g]_{\mathbb{H}}), \tag{7.2}$$

with h^L_γ defined in (3.29). Thus, by Lemma 3.10, the map $\boldsymbol{\phi}_g$ extends to a function on $(\partial\mathbb{D}\smallsetminus\{1,g^{-1}1\})\times\{\mathfrak{Re}(s)>1/2\}$ which is continuous in x for every s, and holomorphic in s for every x.

For $s=1$ we drop the s-dependence in the notation and let

$$\begin{aligned} \boldsymbol{L}(x) &:= \boldsymbol{L}(x,1) = L_\phi([x]_{\mathbb{H}}), \\ \boldsymbol{\phi}_g(x) &:= \boldsymbol{\phi}_g(x,1) = \vartheta_g(x)\boldsymbol{L}(gx) - \boldsymbol{L}(x). \end{aligned} \tag{7.3}$$

With a slight abuse of notation, for an inverse branch $h=h_w\in\mathcal{H}_c$ (or $h\in\mathcal{H}$) associated to a matrix $g=g_w\in G$, we write $\boldsymbol{\phi}_h := \boldsymbol{\phi}_g$.

The following lemmas translate the conclusion of Lemmas 3.10 and 3.11 to the maps $\boldsymbol{\phi}_h$.

Lemma 7.1. *For all $h\in\mathcal{H}_c$, we have the following bounds:*

(1) For all $\varepsilon>0$, we have

$$\boldsymbol{\phi}_h(x) - \boldsymbol{\phi}_h(x') \ll (|h'|^{-\varepsilon} + \max_{y\in\{x,x'\}} |y-1|^{-2(1-\rho_\phi)})|x-x'|^{1-\rho_\phi-\varepsilon} \tag{7.4}$$

uniformly for $h\in\mathcal{H}_c$ and $x,x'\in\mathrm{Dom}(h)\smallsetminus\{1\}$.

(2) If ϕ is cuspidal, then

$$\boldsymbol{\phi}_h(x) \ll 1 \qquad (h\in\mathcal{H}_c, x\in\mathrm{Dom}(h)\smallsetminus\{1\}). \tag{7.5}$$

(3) If ϕ is not cuspidal, then writing h in the form $h=h_{w_0^n w_1}$ and letting $\omega=\omega_{w_0}$ and $\iota_{w_0}, \upsilon_{w_0}$ be as defined in (6.2), we have for $s_\phi\neq 1/2$

$$\begin{aligned} \boldsymbol{\phi}_h(x) &= A_\infty(\phi)G_k(s_\phi, 2(\mathfrak{Im}\, x)^{-1}) + B_\infty(\phi)G_k(1-s_\phi, 2(\mathfrak{Im}\, x)^{-1}) \\ &\quad + \iota_{w_0}\vartheta_h(x)\big(A_\omega(\phi)G_k(s_\phi, \upsilon_{w_0}n) + B_\omega(\phi)G_k(1-s_\phi,\upsilon_{w_0}n)\big) + O(1), \end{aligned} \tag{7.6}$$

uniformly in $x\in\mathrm{Dom}(h)$ and $n\geq 0$. For $s_\phi=1/2$ we have

$$\begin{aligned} \boldsymbol{\phi}_h(x) &= D_\infty(\phi)G_k(\tfrac12, 2(\mathfrak{Im}\, x)^{-1}) + \iota_{w_0}\vartheta_h(x)D_\omega(\phi)G_k(\tfrac12,\upsilon_{w_0}n) \\ &\quad + \begin{cases} O(1), & (B_\infty(\phi)=B_\omega(\phi)=0), \\ O(|x-1|^{-1/2}+n^{1/2}) & \textit{(otherwise)} \end{cases} \end{aligned}$$

with the same uniformity.

(4) For $1/2+\varepsilon<\mathfrak{Re}(s)\ll 1$ and $x\in\mathrm{Dom}(h)\smallsetminus\{1\}$ we have

$$\begin{aligned} \boldsymbol{\phi}_h(x,s) \ll \frac{\mathrm{e}^{\frac{\pi}{2}|\mathfrak{Im}(s)|}}{(1+|s|)^{\mathfrak{Re}(s+s_\phi)-\frac52}} \cdot \Big(&|h'|^{\mathfrak{Re}(s)-1}\big(|x-1|^{2\mathfrak{Re}(s)-2} + |x-1|^{\mathfrak{Re}(s)-1-\rho_\phi-\varepsilon}\big) + \\ &+ 1 + |h'|^{\mathfrak{Re}(s)-1} + |h'|^{(\mathfrak{Re}(s)-1-\rho_\phi-\varepsilon)/2}\Big) \end{aligned} \tag{7.7}$$

Proof. Write $h=h_w$, with $w=w_0^n w_1$ as in the statement. By (6.4), the estimate (7.4) follows from Lemma 3.10 for each fixed n. Thus we may assume that $n\geq 2$. As a consequence, we have that h is an inverse branch of kind (ii) and thus $\mathrm{Dom}(h)$ is contained in some arc $[a]$, $a \in \Sigma$. By the analogue of (3.6) for $\boldsymbol{\phi}$, we have

$$\boldsymbol{\phi}_h(x,s) = \vartheta_{g_{w_1}}(x,s)\boldsymbol{\phi}_{g_{w_0}^n}(g_{w_1}x,s) + |g'_{w_0^n}(g_{w_1}x)|^{s-1}\boldsymbol{\phi}_{g_{w_1}}(x,s) \tag{7.8}$$

for all $x \in \mathrm{Dom}(h)$. Also, recall $\boldsymbol{\phi}_{g_{w_0}^n} \equiv 0$ if $\omega_{w_0} = 1$. By Lemma 3.10 and (6.4), we have

$$\begin{aligned}\boldsymbol{\phi}_{g_{w_1}}(x) - \boldsymbol{\phi}_{g_{w_1}}(x') &\ll \min(|x-1|,|x'-1|)^{-2\varepsilon}\big|[x]_{\mathbb{H}} - [x']_{\mathbb{H}}\big|^{1-\rho_\phi-\varepsilon} \\ &\ll \min(|x-1|,|x'-1|)^{-(2-2\rho_\phi)}|x-x'|^{1-\rho_\phi-\varepsilon}\end{aligned}$$

whereas by Lemma 3.11 and (6.3)-(6.4), we have

$$\Big|\boldsymbol{\phi}_{g_{w_0}^n}(g_{w_1}x) - \boldsymbol{\phi}_{g_{w_0}^n}(g_{w_1}x')\Big| \ll n^\varepsilon|x-x'|^{1-\rho_\phi-\varepsilon}$$

since $\mathrm{Dom}(h) \subseteq \mathrm{Dom}(h_{w_1})$ and so $|g_{w_1}y - 1| \gg 1$ for $y \in \mathrm{Dom}(h)$. Also, as implied by Lemma 9.2 below, the map $x \mapsto \vartheta_{g_{w_1}}(x)$ is constant in $\mathrm{Dom}(h) \smallsetminus \{1\}$. Thus, using the bound $\big|\vartheta_{g_{w_1}}(x)\big| \leq 1$ and taking finite differences, we get

$$\boldsymbol{\phi}_h(x) - \boldsymbol{\phi}_h(x') \ll (n^\varepsilon + \min(|x-1|,|x'-1|)^{-(2-2\rho_\phi)})|x-x'|^{1-\rho_\phi-\varepsilon},$$

which implies the bound (7.4) since $|h'| \asymp (1+n)^{-2}$.

Next assume that ϕ is cuspidal. Then, the bound (7.5) follows as above by the bounds (3.32)-(3.35) and (6.3)-(6.4).

Finally we focus on (7.6). By Lemma 3.11, with $p = [g_{w_0}]_{\mathbb{H}}$, and (6.3), we get

$$\boldsymbol{\phi}_{g_{w_0}^n}(g_{w_1}x) = \iota_{w_0}(\operatorname{sgn} n)^k\vartheta_{g_{w_0}^n}(g_{w_1}x)\big(A_\omega(\phi)G_k(s_\phi,n) + B_\omega(\phi)G_k(1-s_\phi,n)\big) + O(1).$$

On the other hand, by Lemma 3.10 with $\gamma = [g_{w_1}]_{\mathbb{H}}$ and (6.3)-(6.4), we obtain

$$\boldsymbol{\phi}_{g_{w_1}}(x) = A_\omega(\phi)G_k(s_\phi,[x]_{\mathbb{H}}) + B_\omega(\phi)G_k(1-s_\phi,[x]_{\mathbb{H}})\big) + O(1).$$

Inserting these estimates in (7.8), the claimed bound (7.6) follows since $[x]_{\mathbb{H}} = 2(\mathfrak{Im}\,x)^{-1} + O(1)$.

Finally, (7.7) follows from (3.30)-(3.34) as above. We remark that we also have to account for the extra factors $|p'_{\mathbb{H}}(x)|^{1-s}|g'(x)|^{s-1}$ in (7.2), which can be dealt with by using that by (6.3)-(6.4) for $x \in \mathrm{Dom}(h)$ we have

$$|p'_{\mathbb{H}}(x)| \asymp |x-1|^{-2}, \quad |p'_{\mathbb{H}}(g_{w_1}x)| \asymp 1, \quad |g'_{w_1}(x)| \asymp 1, \quad |g'_{w_0^n}(g_{w_1}(x))| \asymp (1+n)^{-2}. \quad \square$$

Remark 7.2. In the non-cuspidal case, the bound (7.7) could be slightly refined, by using the log-factor in (3.3) instead of bounding them by arbitrarily small powers of y. Assuming for simplicity $|\mathfrak{Re}(s-1)| < 1/2-\varepsilon$ and recalling (2.9), this allows to replace the factors $|h'|^{-\varepsilon/2}$ and $|x-1|^{-\varepsilon}$ in the bound by $\log(2+|h'|^{-1})^{(\beta_\phi-1)/2}$ and $\log(2+|x-1|^{-1})^{(\beta_\phi-1)/2}$ respectively.

We are now able to verify the main conditions in [BDLb] on the norm of the period functions at parabolic elements. As in [BDLb], we let

$$\|f\|_{\infty,X} := \sup_{x\in X}|f(x)|, \qquad \|f\|_{(\kappa),X} := \sup_{\substack{x,x'\in X\\ x\neq x'}}\frac{|f(x)-f(x')|}{|x-x'|^\kappa}, \quad (\kappa > 0).$$

Corollary 7.3. *We have*

$$\sum_{\substack{h_1,h_2\in\mathcal{H}_c\\ \mathrm{Im}(h_2)\subset\mathrm{Dom}(h_1)}} |h_1'||h_2'|\|\phi_{h_1}\|^{\alpha}_{\infty,\mathrm{Im}(h_2)} < \infty \qquad \begin{cases} \text{for all } \alpha\geq 0 & (\phi \text{ cuspidal}),\\ \text{for } 0\geq\alpha<2 & (\text{otherwise}),\end{cases} \tag{7.9}$$

$$\sum_{\substack{h_1,h_2,h_3\in\mathcal{H}_c\\ \mathrm{Im}(h_2)\subset\mathrm{Dom}(h_1)\\ \mathrm{Im}(h_3)\subset\mathrm{Dom}(h_2)}} |h_1'||h_2'||h_3'|\|\phi_{h_1}\|_{\infty,\mathrm{Im}(h_2)}\|\phi_{h_2}\|_{\infty,\mathrm{Im}(h_3)} < \infty \tag{7.10}$$

$$\sum_{\substack{h_1,h_2\in\mathcal{H}_c\\ \mathrm{Im}(h_2)\subset\mathrm{Dom}(h_1)}} |h_1'||h_2'|\|\phi_{h_1}\|^{\lambda}_{(1/3),\mathrm{Im}(h_2)} < \infty \qquad \begin{cases} \text{for } 0\leq\lambda<3/2 & (\phi \text{ cuspidal}),\\ \text{for } 0\leq\lambda<6/5 & (\text{otherwise}).\end{cases} \tag{7.11}$$

Proof. By [BDLb, eq. (4.10)] we have $\mathrm{dist}(1,\mathrm{Im}(h_2)) \gg |h_2'|^{1/2}$. Then, using (7.4)-(7.6) we deduce that for each $h_1,h_2\in\mathcal{H}_c$ with $\mathrm{Im}(h_2)\subset\mathrm{Dom}(h_1)$ we have

$$\begin{aligned}\|\phi_{h_1}\|_{(1/3),\mathrm{Im}(h_2)} &\ll |h_1'|^{-\rho_\phi/6-\varepsilon} + \max_{x\in\mathrm{Im}(h_2)} |x-1|^{-2/3-\rho_\phi/3-2\varepsilon}\\ &\ll |h_1'|^{-\rho_\phi/6-\varepsilon} + |h_2'|^{-1/3-\rho_\phi/6-\varepsilon}.\end{aligned}$$

for any $\varepsilon>0$. Thus, for ε small enough and $\lambda<(2/3+\rho_\phi/3)^{-1}$

$$\sum_{\substack{h_1,h_2\in\mathcal{H}_c\\ \mathrm{Im}(h_2)\subset\mathrm{Dom}(h_1)}} |h_1'||h_2'|\|\phi_{h_1}\|^{\lambda}_{(1/3),\mathrm{Im}(h_2)} \ll \sum_{\substack{h_1\in\mathcal{H}_c\\ \mathrm{Im}(h_2)\subset\mathrm{Dom}(h_1)}} (|h_1'||h_2'|)^{1-\lambda(1/3+\rho_\phi/6+\varepsilon)} \ll 1$$

since, by [BDLb, Lemma 5.2] we have $\sum_{h\in\mathcal{H}_c}|h'|^{\sigma}<\infty$ for all $\sigma>\frac12$. This proves (7.11).

Similarly, when ϕ is an Eisenstein series, eq. (7.6) yields

$$\|\phi_{h_1}\|_{\infty,\mathrm{Im}(h_2)} \ll \sup_{x\in\mathrm{Im}(h_2)} (1+|x-1|^{-1/2-\varepsilon} + \left|h_j'\right|^{-1/4-\varepsilon}) \ll |h_1|^{-1/4-\varepsilon} + |h_2'|^{-1/4-\varepsilon}.$$

Thus, we deduce as above that the left-hand side of (7.9) is finite when $\alpha<2$, by taking ε small enough.

When ϕ is cuspidal, by (7.5) we have $\|\phi_{h_1}\|_{\infty,\mathrm{Im}(h_2)} \ll 1$ independently of h_1,h_2, and thus the left-hand side of (7.9) is finite for all $\alpha>0$.

Finally, we move to (7.10) for which a trivial application of (7.6) does not suffice, and we need to be more precise. Given $h=h_{w_0^n w_1}\in\mathcal{H}_c$, denote $\omega_h=\omega_{w_0}$ and $\iota_h=\iota_{w_0}$, so that $\iota_h=0$ if $\omega_h=1$ and $|\iota_h|=1$ otherwise. Since $\mathrm{dist}(1,\mathrm{Im}(h))\gg 1$ for $\omega_h\neq 1$, we have, in any case, $\mathrm{dist}(1,\mathrm{Im}(h))\gg|\iota_h|+|h'|^{1/2}$. Recalling also the factor ι_{w_0} in the second line of (7.6), we then have

$$\|\phi_{h_j}\|_{\infty,\mathrm{Im}(h_{j+1})} \ll 1+\iota_{h_j}|h_j|^{-1/4-\varepsilon} + (1-|\iota_{h_{j+1}}|)\left|h_{j+1}'\right|^{-1/4-\varepsilon}, \qquad j=1,2.$$

Thus, since $(1-|\iota_{h_2}|)|\iota_{h_2}|=0$, we deduce

$$\|\phi_{h_1}\|_{\infty,\mathrm{Im}(h_2)}\|\phi_{h_2}\|_{\infty,\mathrm{Im}(h_3)} \ll (|h_1'||h_2'|)^{-1/4-\varepsilon} + (|h_1'||h_3'|)^{-1/4-\varepsilon} + (|h_2'||h_3'|)^{-1/4-\varepsilon}$$

and the result follows by summing over h_2 and h_3. $\square$

In the rest of the proof, given $z\in\mathbb{C}$ we let $\vec z=\binom{\mathfrak{Re}(z)}{\mathfrak{Im}(z)}\in\mathbb{R}^2$. Also, we let

$$\langle z,w\rangle := \mathfrak{Re}(z\overline{w}) \qquad (z,w\in\mathbb{C}) \tag{7.12}$$

denote the usual scalar product of $\vec z$ and $\vec w$.

7.2. Cuspidal case.

First part of the proof of Theorem 2.2. Let $\boldsymbol{t} \in \mathbb{C}$. We wish to estimate the characteristic function

$$\chi_{Q,\mathfrak{b}}(\boldsymbol{t}) := \mathbb{E}_{Q;\Gamma,\mathfrak{b}}(\mathrm{e}^{i\langle \boldsymbol{t}, L_\phi(x)\rangle}).$$

By Lemma 5.2 and (5.5), we have

$$\chi_{Q,\mathfrak{b}}(\boldsymbol{t}) = \mathbb{E}_j \mathbb{E}_{Q\sqrt{n_j};G,\overline{g}_j[\mathfrak{b}]_{\mathbb{D}}}(\mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{L}(x)\rangle}),$$

where now $x \in C(G)$, $\boldsymbol{L}(x)$ is defined in (7.3), and $\mathbb{E}_j$ denotes a sum over j weighted by $\left|C(G, Q\sqrt{n_j}, \overline{g}_j[\mathfrak{b}]_{\mathbb{D}})\right|/|C(\Gamma, Q, \mathfrak{b})|$.

By Proposition 9.1 below, and since the fundamental domain R has at least 8 sides, the multiplier system $\vartheta_g(x)$ satisfies the stability and mixing hypothesis from [BDLb, Section 5.1]. We also have the bounds (7.9) and (7.11) from Corollary 7.3. By [BDLb, Theorem 1.3], we deduce the existence of $t_0 > 0$ such that for $|\boldsymbol{t}| < t_0$ and any j,

$$\mathbb{E}_{Q\sqrt{n_j};G,\overline{g}_j[\mathfrak{b}]_{\mathbb{D}}}(\mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{L}(x)\rangle}) = \exp\left\{(\log Q)(i\langle \boldsymbol{t}, \boldsymbol{\mu}\rangle - \tfrac{1}{2}\vec{\boldsymbol{t}}^T \boldsymbol{\Sigma} \vec{\boldsymbol{t}} + O(|\boldsymbol{t}|^3)) + O(Q^{-\delta} + |\boldsymbol{t}|)\right\}, \tag{7.13}$$

where

$$\mathfrak{d} := \int_K \log|\mathcal{F}_K'(y)|\xi(y)\,\mathrm{d}s(y) = -\lambda'(1), \tag{7.14}$$

$$\boldsymbol{\mu} := \frac{\mathbf{1}_{\Theta=\{1\}}}{\mathfrak{d}} \int_K \Phi_K(y)\xi(y)\,\mathrm{d}s(y), \tag{7.15}$$

and $\boldsymbol{\Sigma}$ is a 2×2 symmetric positive semi-definite matrix which is defined in [BDLb, Lemma 8.6]; in particular, it is independent from j. The map Φ_K is defined from $\boldsymbol{\phi}_h$ as in [BDLb, eq. (5.9)], and $\xi : K \to \mathbb{R}_+$ is the $\mathcal{F}_K$-invariant measure [BDLb, Proposition 3.6], which is smooth and positive on K. We recall also that $\lambda(s)$ was defined in Lemma 6.1 and that $\mathrm{d}s(y)$ denotes the arc-length measure on $\partial\mathbb{D}$.

Since $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are independent of j, $\chi_{Q,\mathfrak{b}}$ also satisfies the asymptotic relation (7.13). By the Berry-Esseen theorem (see *e.g.* [BD22b, Section 9]) along with the Portmanteau theorem, the stated central limit theorem follows, and to complete the proof of Theorem 2.2 it remains only to show that $\boldsymbol{\mu} = 0$ and $\boldsymbol{\Sigma}$ is diagonal, which will be done in Section 8. ☐

7.3. Non-cuspidal case.

Proof of Theorem 2.5. For the proof, we will focus on the case when $s_\phi = 1/2$ and $B_{\mathfrak{b}}(\phi) = 0$ for all $\mathfrak{b} \in C(\Gamma)$, and will explain afterwards the modifications required to treat the complementary cases. The proof of Theorem 2.5 is similar to that of the cuspidal case, except that now (7.9) holds for $\alpha < 2$ only. By [BDLb, Theorem 1.3], we deduce

$$\chi_{Q,\mathfrak{b}}(\boldsymbol{t}) = \exp\left\{(\log Q)\Big(\frac{1}{\mathfrak{d}}\frac{1}{|\Theta|} I(\boldsymbol{t}) + O(|\boldsymbol{t}|^2)\Big) + O(Q^{-\delta} + |\boldsymbol{t}|)\right\}, \tag{7.16}$$

where, in view of [BDLb, Lemma 8.4] and (7.10),

$$\begin{aligned} I(\boldsymbol{t}) &:= \int_{\partial\mathbb{D}\times\Theta} (\mathrm{e}^{i\langle \boldsymbol{t}, v\Phi_c(x)\rangle} - 1)\varphi(x)\,\mathrm{d}s(x)\,\mathrm{d}v \\ &= i\langle \boldsymbol{t}, \boldsymbol{\mu}\rangle + o(|\boldsymbol{t}|), \end{aligned} \tag{7.17}$$

with $\boldsymbol{\mu}$ defined in (7.15). Here φ is a positive piecewise smooth function on $\partial\mathbb{D}$ which is smooth at vertices of R, see [BDLb, (3.19)], and the map Φ_c is defined by

$$\Phi_c(x) := \vartheta_h(y)^{-1}\boldsymbol{\phi}_h(y), \qquad (y = h(x), h \in \mathcal{H}_c). \tag{7.18}$$

The sets $\mathrm{Im}(h)$, for $h \in \mathcal{H}_c$, partition the circle $\partial\mathbb{D}$ into intervals, and by changing variables (note that $\mathrm{d}s[h(y)] = |h'(y)|\,\mathrm{d}s(y)$), we get

$$I(\boldsymbol{t}) = \sum_{h\in\mathcal{H}_c} I_h(\boldsymbol{t}), \qquad I_h(\boldsymbol{t}) := \int_{\mathrm{Dom}(h)\times\Theta} (\mathrm{e}^{i\langle \boldsymbol{t}, v\boldsymbol{\phi}_h(y)\rangle} - 1)\varphi(h(y))|h'(y)|\,\mathrm{d}s(y)\,\mathrm{d}v. \tag{7.19}$$

Recall the classification of elements of $\mathcal{H}_c$ in Section 6.1. Accordingly, we divide the sum over inverse branches as

$$I(\boldsymbol{t}) = \sum_{b\in\Sigma} I_{h_b}(\boldsymbol{t}) + \sum_{w_0,w_1}\sum_n I_{h_{w_0^n w_1}}(\boldsymbol{t}) \tag{7.20}$$

where n is summed over $\mathbb{Z}_{\geq 1}$ if w_1 has length 1 and over $\mathbb{Z}_{\geq 0}$ otherwise. Also, we recall that for inverse branches of the second kind we have that $\mathrm{Dom}(h_{w_0^n w_1}) = \mathrm{Dom}(h_{w_0 w_1})$ is independent on n.

We wish to apply Lemma 4.1 to estimate these integrals. By Lemma 7.1, the map $\boldsymbol{\phi}_h$ is continuous on $\mathrm{Dom}(h)$, with the exception of a possible blow-up at 1 if $1 \in \mathrm{Dom}(h)$. This has the shape $x \mapsto A_\infty(\phi)G_k(\frac{1}{2}, 2(\mathfrak{Im}\, x)^{-1}) = A_\infty(\phi)G_k(\frac{1}{2}, \pm 2)|x-1|^{-1/2} + o(1)$ as $\partial\mathbb{D} \ni x \to 1$ with $\mathrm{sgn}(\mathfrak{Im}\, x) = \pm 1$ and with G_k defined in (3.31). Note that the coefficients here are independent of h.

Let us first assume that $\Theta \subseteq \{1, -1\}$. We abbreviate

$$\lambda_\infty(\pm) := A_1(\phi)G_k(\tfrac{1}{2}, \pm 2), \qquad \lambda(w_0) := \mathbf{1}_{\omega_{w_0}\neq 1} A_{\omega_{w_0}}(\phi)G_k(\tfrac{1}{2}, \upsilon_{w_0}). \tag{7.21}$$

Combining the facts above, we apply Lemma 4.1 to $\sum_n I_{h_{w_0^n w_1}}(\boldsymbol{t})$, where the integration is taken over the arc $\mathrm{Dom}(h_{w_0 w_1})$ (cf. Remark 4.2). We let $t := |\boldsymbol{t}| \neq 0$ and $x_0 = 1$. By Lemma 7.1 and [BDLb, Lemma 4.5], the hypotheses (i) of Lemma 4.1 are satisfied with the quantities α and β given by

$$\begin{aligned}\alpha_{w_0,w_1} &= \lim_{n\to\infty} n^2 \int_{\mathrm{Dom}(h_{w_0 w_1})} \varphi(h_{w_0^n w_1}(y))\left|h'_{w_0^n w_1}(y)\right| \mathrm{d}s(y)\,\mathrm{d}v \\ &= \varphi(\omega_{w_0}) \lim_{n\to\infty} n^2 \int_{\mathrm{Dom}(h_{w_0 w_1})} \left|h'_{w_0^n w_1}(y)\right| \mathrm{d}s(y),\end{aligned}$$

where ω_{w_0} is the fixed point of g_{w_0},

$$\beta_{w_0,w_1,\pm} = \mathbf{1}_{1\in\overline{\mathrm{Dom}(h_{w_0 w_1})}} \sum_n \varphi(h_{w_0^n w_1}(1\pm i0))|h'_{w_0^n w_1}(1)|, \tag{7.22}$$

with $f(1\pm i0) := \lim_{\partial\mathbb{D}\ni x\to 1, \mathrm{sgn}(\mathfrak{Im}(x))=\pm 1} f(x)$, and $a_{w_0,w_1}, d_{w_0,w_1,\pm}$ satisfying

$$\begin{aligned}|a_{w_0,w_1}|^2 &= |\langle \boldsymbol{t}/|\boldsymbol{t}|, \lambda(w_0)\rangle|^2, \\ |d_{w_0,w_1,\pm}|^2 &= |\langle \boldsymbol{t}/|\boldsymbol{t}|, \lambda_\infty(\pm)\rangle|^2,\end{aligned} \tag{7.23}$$

where we used the assumption $\Theta \subseteq \{1, -1\}$. We deduce that

$$\sum_n I_{h_{w_0^n w_1}}(\boldsymbol{t}) = i\langle \boldsymbol{t}, \vec{\boldsymbol{\mu}}_{w_0,w_1}\rangle - (K^{(1)}_{w_0,w_1}(\boldsymbol{t}) + K^{(2)}_{w_0,w_1}(\boldsymbol{t}))|\boldsymbol{t}|^2|\log|\boldsymbol{t}|| + O(|\boldsymbol{t}|^2), \tag{7.24}$$

for some $\boldsymbol{\mu}_{w_0,w_1} \in \mathbb{C}$ and

$$\begin{aligned}K^{(1)}_{w_0,w_1}(\boldsymbol{t}) &= 2\alpha_{w_0,w_1}\left|\left\langle \boldsymbol{t}/|\boldsymbol{t}|, \lambda(w_0)\right\rangle\right|^2, \\ K^{(2)}_{w_0,w_1}(\boldsymbol{t}) &= 2\sum_n \sum_{1\pm i0\in\mathrm{Dom}(h_{w_0 w_1})} \left|\left\langle \boldsymbol{t}/|\boldsymbol{t}|, \lambda_\infty(\pm)\right\rangle\right|^2 \varphi(h_{w_0^n w_1}(1\pm i0))|h'_{w_0^n w_1}(1)|.\end{aligned}$$

We will now show that $\sum_{w_0,w_1}(K^{(1)}_{w_0,w_1}(\boldsymbol{t})+K^{(2)}_{w_0,w_1}(\boldsymbol{t}))$ is a positive constant independent on $\boldsymbol{t}$. We start by rewriting α_{w_0,w_1} in a more suitable shape. By [BDLb, Lemma 4.5], we have

$$|h'_{w_0^n w_1}(y)| \sim \frac{C(w_0)|h'_{w_1}(y)|}{n^2\operatorname{dist}(h_{w_1}(y),\omega_{w_0})^2}, \qquad (n\to\infty)$$

where $C(w_0)>0$ satisfies $C(\widehat{w_0})=C(w_0)$ and dist is the Euclidean distance. It follows that

$$\alpha_{w_0,w_1}=\varphi(\omega_{w_0})C(w_0)J(w_0,w_1),$$

with

$$J(w_0,w_1):=\int_{\operatorname{Dom}(h_{w_0w_1})}\frac{\left|h'_{w_1}(y)\right|\operatorname{ds}(y)}{\operatorname{dist}(h_{w_1}(y),\omega_{w_0})^2}=\int_{h_{w_1}(\operatorname{Dom}(h_{w_0w_1}))}\frac{\operatorname{ds}(y)}{\operatorname{dist}(y,\omega_{w_0})^2}, \tag{7.25}$$

after the change of variables $y\leftarrow h_{w_1}(y)$. Since $h'_{w_0}(\omega_{w_0})=1$, a quick computation reveals that for all $y\in\partial\mathbb{D}\smallsetminus\{\omega_{w_0}\}$,

$$\frac{\left|h'_{w_0}(y)\right|}{\operatorname{dist}(h_{w_0}(y),\omega_{w_0})^2}=\frac{1}{\operatorname{dist}(y,\omega_{w_0})^2},$$

thus, after a further change of variables, $y\mapsto h_{w_0}(y)=h^{-1}_{\widehat{w_0}}(y)$, we have

$$J(w_0,w_1)=\int_{h_{\widehat{w_0}}\circ h_{w_1}(\operatorname{Dom}(h_{w_0w_1}))}\frac{\operatorname{ds}(y)}{\operatorname{dist}(y,\omega_{w_0})^2}. \tag{7.26}$$

Now, for $w_0=b_1\cdots b_\ell$ and $w_1=b_1\cdots b_r$ with $1\le r\le\ell$, we have

$$h_{\widehat{w_0}}\circ h_{w_1}(\operatorname{Dom}(h_{w_0w_1}))=h_{w_1'}(\operatorname{Dom}(h_{\widehat{w}_0w_1'}))$$

where $w_1'=\widehat{b}_n\cdots\widehat{b}_{r+1}$ is a (possibly empty) prefix of $\widehat{w}_0$. Since $\omega_{w_0}=\omega_{\widehat{w}_0}$, then (7.25)-(7.26) imply $J(w_0,w_1)=J(\widehat{w}_0,w_1')$ and repeating the same change of variable in (7.26) gives that also $J(w_0,w_0)=J(\widehat{w}_0,\widehat{w}_0)$. In particular, $J(w_0,w_1)=J(\widehat{w}_0,w_1^*)$, where $w_1^*:=w_1'$ if $w_1\neq w_0$ and $w_0^*=\widehat{w}_0$. The map $(w_0,w_1)\mapsto(\widehat{w}_0,w_1^*)$ is a permutation among the admissible indexes, thus we obtain

$$\begin{aligned}\sum_{w_0,w_1}K^{(1)}_{w_0,w_1}(\boldsymbol{t})&=\frac12\sum_{w_0,w_1}(K^{(1)}_{w_0,w_1}(\boldsymbol{t})+K^{(1)}_{\widehat{w}_0,w_1^*}(\boldsymbol{t}))\\&=\sum_{w_0,w_1}\varphi(\omega_{w_0})C(w_0)J(w_0,w_1)\big(|\langle\boldsymbol{t}/|\boldsymbol{t}|,\lambda(w_0)\rangle|^2+|\langle\boldsymbol{t}/|\boldsymbol{t}|,\lambda(\widehat{w}_0)\rangle|^2\big).\end{aligned}$$

As observed after its definition (6.2), we have $\upsilon_{\widehat{w}_0}=-\upsilon_{w_0}$. By (3.38) and (7.21), this implies that $\lambda(\widehat{w}_0)=\pm i\lambda(w_0)$ and so, recalling (7.12),

$$|\langle\boldsymbol{t}/|\boldsymbol{t}|,\lambda(w_0)\rangle|^2+|\langle\boldsymbol{t}/|\boldsymbol{t}|,\lambda(\widehat{w}_0)\rangle|^2=\left|\boldsymbol{t}/|\boldsymbol{t}|\cdot\overline{\lambda(w_0)}\right|^2=|\lambda(w_0)|^2.$$

Thus,

$$\sum_{w_0,w_1}K^{(1)}_{w_0,w_1}(\boldsymbol{t})=\sum_{\substack{w_0,w_1\\ \omega_{w_0}\neq1}}\varphi(\omega_{w_0})C(w_0)J(w_0,w_1)\left|A_{\omega_{w_0}}(\phi)G_k(\tfrac12,1)\right|^2=:c^{(1)}, \tag{7.27}$$

say. Notice that all factors in the sum other than $\left|A_{\omega_{w_0}}(\phi)\right|^2$ are strictly positive; also, as w_0 varies, ω_{w_0} spans the set of vertices of the fundamental domain. Thus, $c^{(1)}>0$ unless $A_\omega(\phi)=0$ for all cusp ω not equivalent to 1, in which case $c^{(1)}=0$.

We now consider $\sum_{w_0,w_1} K^{(2)}_{w_0,w_1}$. By (6.5) we can use the permutation $(w_0,w_1)\mapsto(\overline{w_0},\overline{w_1})$ given by complex conjugation and obtain

$$\sum_{w_0,w_1} K^{(2)}_{w_0,w_1}(\boldsymbol{t}) = \frac{1}{2}\sum_{w_0,w_1}\big(K^{(2)}_{w_0,w_1}(\boldsymbol{t}) + K^{(2)}_{\overline{w_0},\overline{w_1}}(\boldsymbol{t})\big).$$

Since φ is real valued and symmetric with respect to the real axis, after the change of variable $\pm\mapsto\mp$, we have

$$K^{(2)}_{\overline{w_0},\overline{w_1}}(\boldsymbol{t}) = 2\sum_n |h'_{w_0^n w_1}(1)| \sum_{1\pm i0\in\mathrm{Dom}(h_{w_0w_1})} |\langle \boldsymbol{t}/|\boldsymbol{t}|, \lambda_\infty(\mp)\rangle|^2 \varphi(h_{w_0^n w_1}(1\pm i0))$$

and thus we have

$$\begin{aligned}\sum_{w_0,w_1} K^{(2)}_{w_0,w_1}(\boldsymbol{t}) &= \sum_{w_0,w_1,n} |h'_{w_0^n w_1}(1)| \sum_{1\pm i0\in\mathrm{Dom}(h_{w_0w_1})} \varphi(h_{w_0^n w_1}(1\pm i0)) \\ &\qquad\qquad \times \big(|\langle \boldsymbol{t}/|\boldsymbol{t}|, \lambda_\infty(+)\rangle|^2 + |\langle \boldsymbol{t}/|\boldsymbol{t}|, \lambda_\infty(-)\rangle|^2\big) \\ &= |A_1(\phi)|^2 \sum_{w_0,w_1,n} |h'_{w_0^n w_1}(1)| \sum_{1\pm i0\in\mathrm{Dom}(h_{w_0w_1})} \varphi(h_{w_0^n w_1}(1\pm i0)) \big|G_k(\tfrac{1}{2},2)\big|^2 \\ &=: c^{(2)},\end{aligned}$$

say, where we used (3.38) and (7.21) as above. Also, $c^{(2)}>0$ if $A_1(\phi)\neq 0$ and $c^{(2)}=0$ otherwise. Recalling (7.27), we then have

$$\sum_{w_0,w_1} (K^{(1)}_{w_0,w_1}(\boldsymbol{t}) + K^{(2)}_{w_0,w_1}(\boldsymbol{t})) = c \tag{7.28}$$

where $c := c^{(1)} + c^{(2)} > 0$ since, by assumption, ϕ is non-cuspidal with $B_\omega(\phi)=0$ for all ω, and thus we must have $A_{\omega'}(\phi)\neq 0$ for some cusp ω'.

Next, we turn to the first summand in (7.20). Applying Lemma 4.1 as above to $I_{h_b}(\boldsymbol{t})$ (with $\alpha_b := 0$, cf. Remark 4.2), we obtain

$$I_{h_b}(\boldsymbol{t}) = i\langle \boldsymbol{t}, \boldsymbol{\mu}_b\rangle - K_b(\boldsymbol{t})|\boldsymbol{t}|^2|\log|\boldsymbol{t}|| + O(|\boldsymbol{t}|^2), \tag{7.29}$$

for some $\boldsymbol{\mu}_b$ and

$$K_b(\boldsymbol{t}) := 2\sum_{1\pm i0\in\mathrm{Dom}(h_b)} \varphi(h_b(1\pm i0))|h_b'(1)||\langle \boldsymbol{t}/|\boldsymbol{t}|, \lambda_\infty(\pm)\rangle|^2.$$

Proceeding[6] as for $K^{(2)}_{w_0,w_1}$, we deduce $\sum_{b\in\Sigma} K_b(\boldsymbol{t}) = d$ for some $d\geq 0$. By (7.20), (7.24), (7.28) and (7.29) and by (7.17), we then have

$$I(\boldsymbol{t}) = i\langle \boldsymbol{t}, \boldsymbol{\mu}\rangle - \frac{\sigma_\phi^2}{2}|\boldsymbol{t}|^2|\log|\boldsymbol{t}|| + O(|\boldsymbol{t}|^2), \tag{7.30}$$

with $\sigma_\phi := \sqrt{2(c+d)} > 0$. Inserting this estimate in (7.16), we conclude using the Berry-Esseen theorem (see [BD22b, Section 9]) that (2.10) holds, modulo the fact that $\boldsymbol{\mu}=0$.

The case $|\Theta|\geq 3$ is similar and in fact simpler. Indeed, we can apply Lemma 4.1 as above with the difference that now (7.23) is to be replaced by

$$\begin{aligned}|a_{w_0,w_1}|^2 &= |\langle \boldsymbol{t}/|\boldsymbol{t}|, v\lambda(w_0)\rangle|^2, \\ |d_{w_0,w_1,\pm}|^2 &= |\langle \boldsymbol{t}/|\boldsymbol{t}|, v\vartheta_{w_0}\lambda_\infty(\pm)\rangle|^2,\end{aligned}$$

[6]In fact, in this case one could also avoid using (6.5), using instead the continuity of φ at $h_b(1)$ and that $1+i0\in\mathrm{Dom}(h_b)$ if and only if $1-i0\in\mathrm{Dom}(h_b)$.

where v is the variable of integration over Θ in (7.19) and ϑ_{w_0} is the value taken by $\vartheta(x)$ in the interior of $\mathrm{Dom}(h_{w_0w_1})$ (see Lemma 9.2 below). Since Θ has order at least 3, then a quick computation shows that for any vectors $\boldsymbol{t}, \kappa \in \mathbb{R}^2$,

$$\int_\Theta |\langle \boldsymbol{t}, v\cdot\kappa\rangle|^2 \,\mathrm{d}v = \|\boldsymbol{t}\|^2\|\kappa\|^2.$$

Thus, performing the integration over v, we obtain (7.30) with a positive σ_ϕ without the need to use the symmetries as done above.

This concludes the proof of Theorem 2.5, save for the equality $\boldsymbol{\mu} = 0$, in the case when $s_\phi = \frac12$ and $B_{\mathfrak{b}} = 0$ for all $\mathfrak{b}$.

In the case when $s_\phi = \frac12$ but $B_{\mathfrak{b}} \neq 0$ for some $\mathfrak{b}$, the previous arguments are adapted with the only changes that Lemma 4.1 is applied with the hypotheses (iii), the logarithmic terms being due to the shape of (3.36). This setting is identical to [BD22b], and leads to an asymptotic estimate of the shape

$$I(\boldsymbol{t}) = i\langle \boldsymbol{t}, \boldsymbol{\mu}\rangle - \frac{\sigma_\phi^2}{2}|\boldsymbol{t}|^2|\log|\boldsymbol{t}||^3 + O(|\boldsymbol{t}|^2|\log|\boldsymbol{t}||^{2+\varepsilon}) \tag{7.31}$$

instead of (7.30). The rest of the proof is identical to the previous case.

Finally, let us focus on the case when $s_\phi = \frac12 + it_\phi \neq \frac12$. We split $I(\boldsymbol{t})$ as in (7.20). For $h = h_{w_0^n w_1} \in \mathcal{H}_c$ and $v \in \Theta$, Lemma 7.1 yields for $\langle \boldsymbol{t}/|\boldsymbol{t}|, v\boldsymbol{\phi}_h(x)\rangle$ an expansion of type

$$\sum_{j=1}^{2} \big(a_j|\mathfrak{Im}\, x|^{-1/2}\cos(t_\phi \log|\mathfrak{Im}\, x| + \theta_j) + d_j|\boldsymbol{t}||n|^{1/2}\cos(t_\phi\log n + \theta_j')\big)$$

where θ_j, θ_j' could depend on $\boldsymbol{t}, v$. The coefficients a_j, d_j are of the shape

$$a_1 = |A_\infty(\phi)G_k(s_\phi, 2\eta_{1,1})|, \qquad a_2 = |B_\infty(\phi)G_k(1-s_\phi, 2\eta_{2,1})|,$$
$$d_1 = \mathbf{1}_{\omega_{w_0}\neq 1}\big|A_{\omega_{w_0}}(\phi)G_k(s_\phi, \eta_{1,2})\big| \qquad d_2 = \mathbf{1}_{\omega_{w_0}\neq 1}\big|B_{\omega_{w_0}}(\phi)G_k(1-s_\phi, \eta_{2,2})\big|$$

where $\eta_{i,j}$ are ±1. The coefficients a_j, d_j do not depend on $\boldsymbol{t}$ (contrary to the earlier case (7.23)), and since $G_k(*,*) \neq 0$, at least one of a_j, d_j is non-zero for some cuspidal cycle w_0. We apply Lemma 4.1 with hypotheses (ii) and obtain

$$\sum_{w_0,w_1}\sum_n I_{h_{w_0^n w_1}}(\boldsymbol{t}) = i\langle \boldsymbol{t}, \boldsymbol{\mu}'\rangle - c|\boldsymbol{t}|^2|\log(|\boldsymbol{t}|)| + O(|\boldsymbol{t}|^2),$$

with $c > 0$ and some $\boldsymbol{\mu}'$. The contribution of $\sum_{a\in\Sigma} I_{h_a}(\boldsymbol{t})$ can be analyzed in the same way, and thus we obtain (7.30), and hence Theorem 2.5, also in this case. ☐

8. Computation of the first and second moments

8.1. **Overview.** In order to complete the proof of Theorems 2.2 and 2.5, we are left to prove :

— that in both the cuspidal and the Eisenstein cases, the mean-value $\boldsymbol{\mu}$ is zero,

— that in the cuspidal case, the variance $\boldsymbol{\Sigma}$ is equal to $\frac12\sigma_\phi^2 I_2$ where I_2 is the identity and σ_ϕ is given by (2.8).

In the modular group case $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ treated in [BD22b], this was done by computing the empirical expectations $\mathbb{E}_Q[\boldsymbol{L}(x)]$, $\mathbb{E}_Q[\boldsymbol{L}(x)^2]$ and $\mathbb{E}_Q[|\boldsymbol{L}(x)|^2]$ directly. The computations made crucial use of the existence, for $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, of an explicit expression for sums

$$\sum_{x\in\Omega_Q} \mathrm{e}(\ell x), \qquad (\ell \in \mathbb{Z}) \tag{8.1}$$

in terms of Ramanujan sums. The Ramanujan sums enjoy a large amount of cancellation and a multiplicative behaviour, related to the existence of Hecke operators, which was crucial in the arguments in [BD22b]. In contrast, for arbitrary groups Γ, existing results on the estimation of (8.1) rely on the analytic properties of the Selberg zeta function (see [PR18, eq. (3.9)]), and the available uniformity in ℓ is prohibitive.

The dynamical systems approach naturally yields the expressions (7.15) for the mean-value, and an analogous expression for the variance. However they involve the period functions $\boldsymbol{\phi}_g$ whose definition in (7.3), as the continuous extension of a map initially defined on the countable set $C(\Gamma)$, is of little use by itself.

We solve this issue by approximating the function $\boldsymbol{\phi}_g$ by the non-central L-values (7.1) with $s = 1 + \varepsilon$ for $\varepsilon > 0$. This is inspired by the construction in [DN26], see equation (3.13) there. The main point regarding our argument here, is that the maps $\boldsymbol{L}(x, s)$ are now continuous in x by uniform convergence. We insert the expression (7.1) into the definition (7.15) for $\boldsymbol{\mu}$, separate the contributions of $\boldsymbol{L}(gx, s)$ and $\boldsymbol{L}(x, s)$, and recombine them using the invariance of ξ. We essentially arrive at an equality of the shape

$$\boldsymbol{\mu} = \lim_{s\to 1^+} (s-1) \int_K \boldsymbol{L}(y, s)\xi(y)\,\mathrm{d}s(x), \tag{8.2}$$

where the factor $s - 1$ arises from the effect of the factor $|g'(x)|^{s-1}$ in (7.1). The equality (8.2) can be thought of as a continuous analogue of the empirical expectation $\lim_{Q\to\infty} \frac{1}{\log Q}\mathbb{E}_Q[\boldsymbol{L}(x)]$. In a way, the dynamical systems argument took care of replacing the discrete average over $C(\Gamma)$ by a continuous average over $\partial\mathbb{D}$. The continuous average (8.2) is easily estimated by inserting the Fourier expansion stemming from (2.6), using a rough bound on a_n and on the Fourier coefficients of ξ, and this leads to the value $\boldsymbol{\mu} = 0$.

Regarding the variance $\boldsymbol{\Sigma}$, although the computations are more involved, we eventually find that the same idea leads to the expression

$$\boldsymbol{\Sigma} = \lim_{s\to 1^+} 2(s-1) \int_K \vec{\boldsymbol{L}}(y, s)\vec{\boldsymbol{L}}(y, s)^T \xi(y)\,\mathrm{d}s(y).$$

Expanding the product leads to sums of the shape $\sum_n a_n a_{n+k}$, which are handled by means of Rankin-Selberg theory. Contrarily to the mean-value, the diagonal terms $k = 0$ survive the integration over y and give the main contribution.

8.2. **Approximation lemmas.**

8.2.1. *Approximating $\boldsymbol{\phi}_h(y)$ by non-central L-values.* Recall the definition of the non-central L-values $\boldsymbol{L}(x, s)$ and period functions $\boldsymbol{\phi}_g(x, s)$ in (7.1). Recall also the definition of ξ_s and $\mathcal{U}$ in Lemma 6.1, of the maps $p_{\mathbb{H}}, p_{\mathbb{D}}$ in (5.1), and that $\mathcal{H}$ denotes the set of inverse branches of $\mathcal{F}_K$ (cf. Section 6.2).

Lemma 8.1. *(i) For all $g \in G$ and $x \in \partial\mathbb{D} \smallsetminus \{1, g^{-1}1\}$, the map $s \mapsto \boldsymbol{\phi}_g(x, s)$ extends to an analytic function on $\mathfrak{Re}(s) > 1/2$.*

(ii) For $\mathfrak{Re}(s) > 3/2$, the map $x \mapsto \boldsymbol{L}(x, s)$ is continuous and bounded on $\mathbb{R}$.

(iii) Uniformly for $s \in \mathcal{U}$ and $\mathfrak{Re}(s) > 1 - \delta$, we have

$$\sum_{h\in\mathcal{H}} \| h' |\boldsymbol{\phi}_h(\cdot, s)|^\alpha \|_{\infty,\mathrm{Dom}(h)} \ll 1, \qquad \big(\alpha < \min(\tfrac{1}{\delta+\rho_\phi}, \tfrac{1}{2\delta})\big) \tag{8.3}$$

Proof. The first two points are proven in Section 7.1. The third points follow from [BDLb, Lemma 6.1] and (7.7). ☐

8.2.2. *Leading eigenparameters.* For $s \in \mathcal{U}$, $x \in \mathbb{R}$, we let

$$\zeta_s(x) = \xi_s\Big(\frac{x-i}{x+i}\Big)\frac{2^s}{(x^2+1)^s}\mathbf{1}_{x\in[K]_{\mathbb{H}}}, \tag{8.4}$$

where we recall $[K]_{\mathbb{H}} = p_{\mathbb{H}}(K)$. Note that, by a change of variable, for all $n \in \mathbb{Z}$ we have

$$\widehat{\zeta_s}(n) := \int_{\mathbb{R}} \zeta_s(x)\mathrm{e}(-nx)\,\mathrm{d}x = \int_K |p'_{\mathbb{H}}(y)|^{1-s}\xi_s(y)\mathrm{e}(-np_{\mathbb{H}}(y))\,\mathrm{ds}(y). \tag{8.5}$$

Lemma 8.2. *We have $\widehat{\zeta_s}(n) \ll n^{-1}$ uniformly for $n \in \mathbb{Z}_{\neq 0}$ and for $s \in \mathcal{U}$.*

Proof. Note that $[K]_{\mathbb{H}}$ is a finite union of bounded intervals. By Lemma 6.1 we have $|\zeta_s(x)|, |\zeta_s'(x)| \ll 1/(x^2+1)^{1-2\delta}$ uniformly for $s \in \mathcal{U}$ and $x \in [K]_{\mathbb{H}}$, whence by partial summation we deduce $\widehat{\zeta_s}(n) \ll 1/n$ with the same uniformity. □

8.3. **First moment.** We recall the expression

$$\boldsymbol{\mu} := \frac{\mathbf{1}_{\Theta=\{1\}}}{\mathfrak{d}}\int_K \Phi_K(y)\xi(y)\,\mathrm{ds}(y)$$

for the mean-value, where $\mathfrak{d}$ is defined in (7.14).

Proposition 8.3. *We have $\boldsymbol{\mu} = 0$.*

Proof. We may assume $\Theta = \{1\}$. For all $s \in \mathcal{U}$, define

$$\boldsymbol{\mu}(s) := \sum_{h\in\mathcal{H}} \frac{1}{\mathfrak{d}}\int_{\mathrm{Dom}(h)} \boldsymbol{\phi}_h(y,s)\xi_s(h(y))|h'(y)|\,\mathrm{ds}(y) \tag{8.6}$$

and note that by Lemma 6.1.(ii) and (8.3) we have

$$\sum_{h\in\mathcal{H}}\int_{\mathrm{Dom}(h)} |\boldsymbol{\phi}_h(y,s)\xi_s(h(y))h'(y)|\,\mathrm{ds}(y) \ll 1.$$

By Lemmas 8.1.(i) and 6.1.(iii), we deduce that the map $s \mapsto \boldsymbol{\mu}(s)$ is analytic in $\mathcal{U}$, and

$$\boldsymbol{\mu} = \boldsymbol{\mu}(1).$$

Assume next $\mathfrak{Re}(s) > 3/2$. We insert the explicit expression from (7.1), and separate both contributions to the integral. Note that $\vartheta_h(y) = 1$ since we assumed $\Theta = \{1\}$. We find for all $\mathfrak{Re}(s) > 3/2$

$$\boldsymbol{\mu}(s) = M_1(s) - M_2(s),$$

where

$$M_1(s) = \sum_{h\in\mathcal{H}} \frac{1}{\mathfrak{d}}\int_{\mathrm{Dom}(h)} \boldsymbol{L}(h(y),s)\xi_s(h(y))|h'(y)|\,\mathrm{ds}(y),$$

$$M_2(s) = \sum_{h\in\mathcal{H}} \frac{1}{\mathfrak{d}}\int_{\mathrm{Dom}(h)} \boldsymbol{L}(y,s)\xi_s(h(y))|h'(y)|^s\,\mathrm{ds}(y).$$

Since $\bigcup_{h\in\mathcal{H}} \mathrm{Im}(h)$ is a partition of K, after a change of variables $M_1(s)$ becomes

$$M_1(s) = \sum_{h\in\mathcal{H}} \frac{1}{\mathfrak{d}}\int_{\mathrm{Im}(h)} \boldsymbol{L}(y,s)\xi_s(y)\,\mathrm{ds}(y) = \frac{1}{\mathfrak{d}}\int_K \boldsymbol{L}(y,s)\xi_s(y)\,\mathrm{ds}(y).$$

Concerning $M_2(s)$, by definition we have for all $y \in K$

$$\sum_{h\in\mathcal{H}, y\in\mathrm{Dom}(h)} |h'(y)|^s\xi_s(h(y)) = \mathscr{L}_s[\xi_s](y) = \lambda(s)\xi_s(y)$$

with $\lambda(s)$ as in Lemma 6.1. We deduce

$$M_2(s) = \frac{\lambda(s)}{\mathfrak{d}} \int_K \boldsymbol{L}(y,s)\xi_s(y)\,\mathrm{d}s(y).$$

We conclude that

$$\boldsymbol{\mu}(s) = \frac{1-\lambda(s)}{\mathfrak{d}} \int_K \boldsymbol{L}(y,s)\xi_s(y)\,\mathrm{d}s(y). \tag{8.7}$$

We insert the definition

$$\boldsymbol{L}(y,s) = |p_{\mathbb{H}}'(y)|^{1-s} \sum_{n\geq 1} a(n)\mathrm{e}(2\pi n p_{\mathbb{H}}(y))n^{1-s}, \tag{8.8}$$

which is allowed by the bound (2.5), since $\mathfrak{Re}(s) > 3/2$. We obtain

$$\boldsymbol{\mu}(s) = \frac{1-\lambda(s)}{\mathfrak{d}} \sum_{n\geq 1} a(n)n^{1-s}\widehat{\zeta}_s(-n), \qquad (\mathfrak{Re}(s) > 3/2), \tag{8.9}$$

where $\widehat{\zeta}_s(\cdot)$ is defined in (8.5). By Lemma 8.2, the sum over n on the right-hand side converges uniformly for $s \in \mathcal{U}$. Therefore, we conclude by analytic continuation that the identity (8.9) holds for all $s \in \mathcal{U}$. Since $\lambda(1) = 1$, the proof is concluded by setting $s = 1$. □

8.4. **Second moment in the cuspidal case.** In the remainder of this section, we restrict s to a real neighborhood of 1. Let $\delta > 0$ be fixed and

$$\mathcal{V} := [1, 1+\delta].$$

For $s \in \mathcal{V}$, $x \in K$ and $\vartheta \in \Theta$, we define Φ_s by

$$\Phi_{K,s}(x,\vartheta) = \vartheta\vartheta_h(y)^{-1}\boldsymbol{\phi}_h(y,s), \qquad (h \in \mathcal{H}, x = h(y)), \tag{8.10}$$

in analogy with (7.18). For $s \in \mathcal{V}$, and picking δ small enough, we let

$$\boldsymbol{\chi}_s = \frac{1}{\xi_s}(\mathrm{Id} - \mathscr{N}_s)^{-1}\mathscr{L}_s[\Phi_{K,s}\xi_{2s-1}(\cdot)], \qquad \boldsymbol{\chi} := \boldsymbol{\chi}_1, \tag{8.11}$$

where $\mathscr{N}_s$ is obtained from $\mathscr{L}_s$ by substracting the spectral projection $\mathscr{P}_s$ onto its leading eigenspace. By [BDLb, Proposition 7.3] and perturbation theory, the operator $\mathscr{N}_s$ is contracting on $H^{1/6}(K \times \Theta)$ for $s \in \mathcal{V}$ for δ small enough, which justifies the definition (8.11).

Lemma 8.4. *(1) The map $\boldsymbol{\chi}_s$ belongs to $H^{1/6}(K \times \Theta)$ for all $s \in \mathcal{V}$.*

(2) For all $s \in \mathcal{V}$ we have

$$\boldsymbol{\chi}_s\xi_s - \mathscr{L}_s[\boldsymbol{\chi}_s\xi_s] = \mathscr{L}_s[\Phi_{K,s}\xi_{2s-1}] - \mathscr{P}_s[\boldsymbol{\chi}_s\xi_s]. \tag{8.12}$$

(3) As $s \to 1^+$, we have $\boldsymbol{\chi}_s \to \boldsymbol{\chi}$ uniformly on K.

Proof. From the definition of $\mathscr{L}_s$ in (6.6), we have

$$\mathscr{L}_s[\Phi_{K,s}\xi_{2s-1}](y,\vartheta_1) = \vartheta_1 \sum_{\substack{h\in\mathcal{H} \\ y\in\mathrm{Dom}(h)}} |h'(y)|^s \boldsymbol{\phi}_h(y,s)\xi_{2s-1}(h(y)). \tag{8.13}$$

By Lemma 6.1 and (8.3), uniformly in $y \in K$ and $\mathfrak{Re}(s) \geq 1$ we have

$$\sum_{\substack{h\in\mathcal{H} \\ y\in\mathrm{Dom}(h)}} \left||h'(y)|^s \boldsymbol{\phi}_h(y,s)\xi_{2s-1}(h(y))\right| \ll \sum_{h\in\mathcal{H}} |h'|^{\mathfrak{Re}(s)} \|\boldsymbol{\phi}_h\|_{\infty,\mathrm{Dom}(h)} \ll 1.$$

Let $h \in \mathcal{H}$, ϑ_1 and $s \in \mathcal{V}$ be fixed . For $y_1, y_2 \in \mathrm{Dom}(h)$, we have

$$|h'(y_1)^s - h'(y_2)^s| \ll |h'| \times |y_1 - y_2|$$

by the bounded distortion property [BDLb, eq. (4.1)]. We also have

$$|\xi_{2s-1}(h(y_1)) - \xi_{2s-1}(h(y_2))| \ll |h'| \times |y_1 - y_2|$$

uniformly in s, by Lemma 6.1.(ii). Therefore, we get that the map

$$y \mapsto |h'(y)|^s \boldsymbol{\phi}_h(y, s)\xi_{2s-1}(h(y))$$

is $\frac{1}{6}$-Hölder continuous on K, with Hölder constant bounded by

$$|h'|(\|\boldsymbol{\phi}_h\|_{\infty,\mathrm{Dom}(h)} + \|\boldsymbol{\phi}_h\|_{(1/6),\mathrm{Dom}(h)}) \ll |h'|(\|\boldsymbol{\phi}_h\|_{\infty,\mathrm{Dom}(h)} + \|\boldsymbol{\phi}_h\|_{(1/3),\mathrm{Dom}(h)}),$$

by [BDLb, eq. (6.4)]. By (7.9) and (7.11) in conjunction with [BDLb, Lemma 6.1] we have that $\sum_{h\in\mathcal{H}} |h'|(\|\boldsymbol{\phi}_h\|_{\infty,} + \|\boldsymbol{\phi}_h\|_{(1/3)}) \ll 1$ and thus the map (8.13) belongs to $H^{1/6}(K \times \Theta)$ as claimed, and therefore so does $\boldsymbol{\chi}_s$. The identity (8.12) is obvious from the definition (8.11). Finally, since ξ_{2s-1}, $\boldsymbol{\phi}_h(\cdot, s)$ and $|h'|^s$ are holomorphic in s, we deduce by uniform convergence that the family of maps (8.13) is holomorphic in $s \in \mathcal{V}$ in the sense of [Kat95, Section VII.1]. By perturbation theory [Kat95, Theorem VII.1.7] and the Perron-Frobenius theorem for $\mathscr{L}_s$ (already used in Lemma 6.1 on another functional space), the family of operators $(\mathscr{N}_s)_{s\in\mathcal{V}}$ is bounded-holomorphic, therefore $s \mapsto \chi_s$ is also holomorphic. This yields the third point. ☐

Lemma 8.5. *For all $\vartheta \in \Theta$, as $s \to 1$ with $s > 1$, we have*

$$\int_K \vec{\boldsymbol{L}}(x, s)\vec{\mathscr{P}}_s[\boldsymbol{\chi}_s\xi_s]^T(x, \vartheta)\,\mathrm{ds}(x) \to 0. \tag{8.14}$$

Proof. Since $\mathscr{P}\mathscr{N} = 0$, so that $\mathscr{P}(\mathrm{Id} - \mathscr{N})^{-1} = \mathscr{P}$, and $\mathscr{P}\mathscr{L} = \mathscr{P}$, we have

$$\mathscr{P}[\boldsymbol{\chi}\xi] = \mathscr{P}[\Phi_K\xi] = \Big(\int_{K\times\Theta} \Phi_K\xi\Big)\xi = 0$$

by Proposition 8.3. Thus we have $\mathscr{P}_s[\boldsymbol{\chi}_s\xi_s] = (s-1)g_s$ for some bounded, holomorphic family of maps $g_s \in H^{1/6}(K\times\Theta)$. Using the complex number representation of $\boldsymbol{L}$ and g_s, the limit (8.14) is equivalent to

$$(s-1)\sum_{n\geq 1} \frac{a_n}{n^{s-1}} \int_K |p'_{\mathbb{H}}(y)|^{1-s}\mathrm{e}(np_{\mathbb{H}}(y))g_s(y, \vartheta)\,\mathrm{ds}(y) \to 0 \qquad (s \to 1) \tag{8.15}$$

and the same with g_s replaced by $\overline{g_s}$. We focus on (8.15). Integrating by parts (see *e.g.* [BD22b, Lemma 4.1]), we deduce

$$\int_K |p'_{\mathbb{H}}(y)|^{1-s}\mathrm{e}(np_{\mathbb{H}}(y))g_s(y, \vartheta)\,\mathrm{ds}(y) \ll n^{-1/6}$$

uniformly for s in a neighborhood of 1. Thus, as in the proof of Proposition 8.3, the sum over n in (8.15) is uniformly bounded, and we conclude that (8.15) holds as claimed. ☐

The covariance matrix in (7.13), which we seek to evaluate, is given explicitly in [BDLb, Lemma 8.6]. An alternative expression, more useful for our purpose, is given in [BDLb, eq. (8.12)] and states

$$\boldsymbol{\Sigma} := \frac{1}{\mathfrak{d}} \int_{K\times\Theta} (\vec{\Phi}_K\vec{\Phi}_K^T + \vec{\boldsymbol{\chi}}\vec{\Phi}_K^T + \vec{\Phi}_K\vec{\boldsymbol{\chi}}^T)\xi\,\mathrm{d}\nu. \tag{8.16}$$

Proposition 8.6. *We have $\boldsymbol{\Sigma} = \frac{1}{2}\sigma_\phi^2\,\mathrm{Id}$, with σ_ϕ given in* (2.8).

Remark 8.7. It is instructive to compare with the value obtained by Nordentoft in [Nor21] in the case when k is even and ϕ is associated to a cuspidal holomorphic form $f(z) = y^{-k/2}\phi(z)$. We then have $s_\phi = k/2$ and $\|\phi\| = \|f\|_k$ the Petersson k-norm, therefore

$$\sigma_\phi^2 = \frac{\|f\|_k^2}{(k-1)!\operatorname{Vol}(\Gamma\backslash\mathbb{H})}$$

which recovers the value obtained in [Nor21, eq. (1.4)]. Note that the factor $(4\pi)^k$ in [Nor21] is due to a difference in normalization originating from the Whittaker function $W_{k/2,(k-1)/2}(4\pi y)$.

Proof of Proposition 8.6. First we reduce to the case $k \in \{0, 1\}$. To this end, let $\psi = L_k\phi$ be the level-lowered form, of weight $k-2$ and eigenvalue $s_\psi = s_\phi$. Then $a_\psi(n) = -a_\phi(n)$ for all $n \geq 0$, so that $L_\psi(x) = -L_\phi(x)$. On the other hand,

$$\frac{\sigma_\psi^2}{\sigma_\phi^2} = \frac{\|\psi\|^2}{\|\phi\|^2}(-s_\phi + \tfrac{k}{2})(-1 + s_\phi + \tfrac{k}{2}) = 1,$$

where the last equality is [DFI02, eq. (4.38)]. When $k \geq 2$, repeating this operation shows that we may replace ϕ in Proposition 8.6 by its "ancestor" of weight 0 or 1. A similar argument holds when $k \leq -1$, using instead the level-raising operator. Thus we may assume $k \in \{0, 1\}$.

We start from the expression (8.16). By Lemmas 6.1, 8.1, 8.4, and uniform convergence, we get

$$\boldsymbol{\Sigma} = \lim_{s\to1^+} \boldsymbol{\Sigma}(s),$$

where

$$\boldsymbol{\Sigma}(s) := \frac{1}{\mathfrak{d}}\int_{K\times\Theta} \Big(\vec{\Phi}_{K,s}\vec{\Phi}_{K,s}^T\xi_{2s-1} + (\vec{\boldsymbol{\chi}}_s\vec{\Phi}_{K,s}^T + \vec{\Phi}_{K,s}\vec{\boldsymbol{\chi}}_s^T)\xi_s\Big)\,\mathrm{d}\nu. \tag{8.17}$$

Recall the definition (6.7), and for $s > 1$, let

$$\boldsymbol{L}_s(x,\vartheta) := \vartheta\boldsymbol{L}(s,x). \tag{8.18}$$

Thus, by (8.10) and (7.1), we may write

$$\Phi_{K,s}(x,\vartheta) = \boldsymbol{L}_s(x,\vartheta) - |\mathcal{F}_K'(x)|^{1-s}(\boldsymbol{L}_s\circ\mathcal{F}_K)(x,\vartheta), \tag{8.19}$$

whence

$$\begin{aligned}\int_{K\times\Theta}\vec{\Phi}_{K,s}\vec{\boldsymbol{\chi}}_s^T\xi_s\,\mathrm{d}\nu &= \int_{K\times\Theta}(\vec{\boldsymbol{L}}_s - |\mathcal{F}_K'|^{1-s}(\vec{\boldsymbol{L}}_s\circ\mathcal{F}_K))\vec{\boldsymbol{\chi}}_s^T\xi_s\,\mathrm{d}\nu \\ &= \int_{K\times\Theta}\vec{\boldsymbol{L}}_s(\vec{\boldsymbol{\chi}}_s\xi_s - \vec{\mathscr{L}}_s[\boldsymbol{\chi}_s\xi_s])^T\,\mathrm{d}\nu,\end{aligned}$$

by (6.9). Using (8.12) and (6.9), this becomes

$$\begin{aligned}\int_{K\times\Theta}\vec{\Phi}_{K,s}\vec{\boldsymbol{\chi}}_s^T\xi_s\,\mathrm{d}\nu &= \int_{K\times\Theta}\vec{\boldsymbol{L}}_s\vec{\mathscr{L}}_s[\Phi_{K,s}\xi_{2s-1}]^T\,\mathrm{d}\nu - \int_{K\times\Theta}\vec{\boldsymbol{L}}_s\vec{\mathscr{P}}_s[\boldsymbol{\chi}_s\xi_s]^T\,\mathrm{d}\nu \\ &= \int_{K\times\Theta}(\vec{\boldsymbol{L}}_s\circ\mathcal{F}_K)\vec{\Phi}_{K,s}^T|\mathcal{F}_K'|^{1-s}\xi_{2s-1}\,\mathrm{d}\nu - \int_{K\times\Theta}\vec{\boldsymbol{L}}_s\vec{\mathscr{P}}_s[\boldsymbol{\chi}_s\xi_s]^T\,\mathrm{d}\nu.\end{aligned}$$

By Lemma 8.5, the second integral is $o(1)$ as $s \to 1^+$. A similar identity holds for $\int \vec{\boldsymbol{\chi}}_s\vec{\Phi}_{K,s}^T\xi_s$. We deduce

$$\boldsymbol{\Sigma}(s) = \frac{1}{\mathfrak{d}}\int_{K\times\Theta}\big(\vec{\Phi}_{K,s}\vec{\Phi}_{K,s}^T + (\vec{\boldsymbol{L}}_s\circ\mathcal{F}_K)\vec{\Phi}_{K,s}^T|\mathcal{F}_K'|^{1-s} + \vec{\Phi}_{K,s}(\vec{\boldsymbol{L}}_s\circ\mathcal{F}_K)^T|\mathcal{F}_K'|^{1-s}\big)\xi_{2s-1}\,\mathrm{d}\nu + o(1).$$

Inserting the expression (8.19) and simplifying, we get

$$\boldsymbol{\Sigma}(s) = \frac{1}{\mathfrak{d}} \int_{K\times\Theta} (\vec{\boldsymbol{L}}_s \vec{\boldsymbol{L}}_s^T - (\vec{\boldsymbol{L}}_s \circ \mathcal{F}_K)(\vec{\boldsymbol{L}}_s \circ \mathcal{F}_K)^T |\mathcal{F}_K'|^{2-2s})\xi_{2s-1}\,\mathrm{d}\nu + o(1),$$

and finally, since $\mathscr{L}_{2s-1}\xi_{2s-1} = \lambda(2s-1)\xi_{2s-1}$ and using (6.9) once again, we arrive at

$$\boldsymbol{\Sigma}(s) = o(1) + \frac{1-\lambda(2s-1)}{\mathfrak{d}} \int_{K\times\Theta} \vec{\boldsymbol{L}}_s \vec{\boldsymbol{L}}_s^T \xi_{2s-1}\,\mathrm{d}\nu, \qquad (s\to 1^+).$$

By definition of $\mathfrak{d}$ [BDLb, eq. (8.3)], we have $1-\lambda(2s-1) \sim 2(s-1)\mathfrak{d}$ as $s\to 1^+$. Thus, we are left to show that

$$\int_{K\times\Theta} \vec{\boldsymbol{L}}_s \vec{\boldsymbol{L}}_s^T \xi_{2s-1}\,\mathrm{d}\nu \sim \frac{\sigma_\phi^2}{4(s-1)}\,\mathrm{Id} \tag{8.20}$$

as $s\to 1^+$. The asymptotic behaviour (8.20) is equivalent to the limits

$$\int_{K\times\Theta} |\boldsymbol{L}_s|^2 \xi_{2s-1}\,\mathrm{d}\nu \sim \frac{\sigma_\phi^2}{2(s-1)}, \qquad \int_{K\times\Theta} \boldsymbol{L}_s^2 \xi_{2s-1}\,\mathrm{d}\nu = o\Big(\frac{1}{s-1}\Big),$$

where here $\boldsymbol{L}_s(x)$ is interpreted as a complex number. Recall the definition (8.18). Note that in the first integral, the Θ integral trivially evaluates to 1 (the integrand is independent from ϑ), while in the second integral, we have $\int_\Theta \vartheta^2 = \mathbf{1}_{\Theta\subset\{\pm1\}}$. Changing variables and using the definition (7.1), we obtain that it suffices to show

$$\int_{\mathbb{R}} |L(\phi, s-1/2, x)|^2 \zeta_s(x)\,\mathrm{d}x \sim \frac{\sigma_\phi^2}{2(s-1)}, \tag{8.21}$$

$$\int_{\mathbb{R}} L(\phi, s-1/2, x)^2 \zeta_s(x)\,\mathrm{d}x = o\Big(\frac{1}{s-1}\Big), \tag{8.22}$$

where we recall $\zeta_s(x)$ is as defined in (8.4).

Let $H\geq 1$ be a parameter. By Vaaler approximation [Vaa85, Theorem 11] (choosing $\delta = H$), there exist two maps ζ_s^* and $\zeta_s^\sharp$ whose Fourier coefficients satisfy

$$\begin{cases} \widehat{\zeta_s^*}(0) = \int \zeta_s, \\ \widehat{\zeta_s^\sharp}(0) \ll 1/H, \\ \widehat{\zeta_s^*}(h), \widehat{\zeta_s^\sharp}(h) \ll 1 & (h\in\mathbb{Z}, s\in[1,2]), \\ \widehat{\zeta_s^*}(h) = \widehat{\zeta_s^\sharp}(h) = 0 & (|h|>H), \end{cases}$$

and

$$|\zeta_s(x) - \zeta_s^*(x)| \leq \zeta_s^\sharp(x) \qquad (x\in\mathbb{R}).$$

We insert this inequality in the integral $\int_{\mathbb{R}} |L(\phi, s-1/2, x)|^2 \zeta_s(x)\,\mathrm{d}x$, replace $L(\phi, s-1/2, x)$ by its Fourier expansion (2.6) and interchange sums and integral, which is allowed for $s>1$ by the Wilton bound [DN26, eq. (2.8)]. We thus obtain

$$\int_{\mathbb{R}} |L(\phi, s-1/2, x)|^2 \zeta_s(x)\,\mathrm{d}x = \Big(\int_{\mathbb{R}} \zeta_s + O(H^{-1})\Big) I_s(0) + O\Big(\sum_{0<|h|\leq H} |I_s(h)|\Big), \tag{8.23}$$

where

$$I_s(h) = \sum_{\substack{m,n\geq 1\\ n-m=h}} a(m)\overline{a(n)}(mn)^{1-s}. \tag{8.24}$$

Note that $\int \zeta_s = \int_K |p'_{\mathbb{H}}|^{1-s}\xi_{2s-1} \to 1$ as $s \to 1$. Letting $H \to \infty$ sufficiently slowly as $s \to 1^+$, we see that to prove the asymptotic behaviour (8.21) it suffices to show that

$$I_s(0) \sim \frac{\sigma_\phi^2}{2(s-1)}, \tag{8.25}$$

$$I_s(h) \ll_h 1, \tag{8.26}$$

where the second bound is uniform in $s \geq 1$.

For $h = 0$ we have

$$I_s(0) = \sum_{n\geq 1} \frac{|a(n)|^2}{n^{2(s-1)}}.$$

We apply the Rankin-Selberg method [Iwa95, eq. (8.10)]. Define

$$I^{\pm}(s) := \sum_{\mathrm{sgn}(n)=\pm} \frac{|a(n)|^2}{|n|^{2(s-1)}},$$

so that $I_s(0) = I^+(s)$. Define further, for all $\ell \in \mathbb{Z}$,

$$C_\ell(s) := \int_0^\infty W_{\ell/2,it_\phi}(y)^2 y^{2(s-1)} \frac{\mathrm{d}y}{y}. \tag{8.27}$$

By the Rankin-Selberg method, we have

$$C_k(s)I^+(s) + C_{-k}(s)I^-(s) = (4\pi)^{2(s-1)} J_\phi(s) \tag{8.28}$$

where

$$J_\phi(s) := \int_{\Gamma\backslash\mathbb{H}} |\phi(z)|^2 E(2s-1, z) \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

and $E(\cdot, z)$ is the standard Eisenstein series at ∞. On the other hand, let $\psi = R_k\phi$ be the level-raised form obtained from ϕ. It is of weight $k+2$ and has a Fourier expansion

$$\psi(z) = \sum_{n\neq 0} a_\psi(n) \mathrm{e}(n\,\mathfrak{Re}(z)) W_{\mathrm{sgn}(n)\frac{k+2}{2},it_\phi}(4\pi n\,\mathfrak{Im}(z)),$$

where

$$a_\psi(n) = \begin{cases} -a(n) & (n \geq 1), \\ (s_\phi + \frac{k}{2})(1 - s_\phi + \frac{k}{2})a(n) & (n \leq -1). \end{cases}$$

By the Rankin-Selberg method again, we find

$$C_{k+2}(s)I^+(s) + \left|(s_\phi + \tfrac{k}{2})(1 - s_\phi + \tfrac{k}{2})\right|^2 C_{-k-2}(s)I^-(s) = (4\pi)^{2(s-1)} J_\psi(s). \tag{8.29}$$

The maps C_ℓ are regular at $s = 1$, since $W_{\ell/2,it_\phi}(y) \ll y^{1/2-|\mathfrak{Re}(s_\phi)|}$ in the neighborhood of 0. The value $C_\ell(1)$ is computed using the formula in [GR07, (7.611.4)][7]. We obtain

$$C_\ell(1) = \frac{1}{\sin(2\pi s_\phi)} \prod_{1\leq j\leq|\ell|} (s_\phi + \tfrac{|\ell|}{2} - j)^\eta \Big(\eta \sin(\pi(s_\phi + \tfrac{\ell}{2})) \sum_{1\leq j\leq|\ell|} \frac{1}{s_\phi + \frac{|\ell|}{2} - j} + \pi\cos(\pi(s_\phi + \tfrac{\ell}{2}))\Big) \tag{8.30}$$

where $\eta = \mathrm{sgn}(\ell) \in \{-1, 1\}$ if $\ell \neq 0$ and $\eta = 0$ if $\ell = 0$.

[7] The related formula [DFI02, eq. (19.6)] should be corrected by a non-constant factor when $k \neq 0$. This is due to the fact that [GR07, (3.384.9)], which is used there, is not immediately applicable when $\mathfrak{Re}(s_\phi) = 1/2$ due to lack of absolute convergence. We use instead [GR07, (7.611.4)] which is proven with the arguments described in [Buc53, pp. 112–116].

Let $\mu_k = (s_\phi + \frac{k}{2})(1 - s_\phi + \frac{k}{2})$. We solve the equations (8.28) and (8.29) for $I^+(s)$, which is possible since

$$\Delta_k(s) = \mu_k^2 C_k(s) C_{-k-2}(s) - C_{-k}(s) C_{k+2}(s) = \frac{\pi(2s_\phi - 1)}{\sin(2\pi s_\phi)} \neq 0$$

for $k \in \{0, 1\}$. We find

$$I^+(s) = \frac{1}{\Delta_k(s)}(\mu_k^2 C_{-k-2}(s) J_\phi(s) - C_{-k}(s) J_\psi(s)).$$

By [Iwa02, Proposition 6.13], we have

$$J_\phi(s) \sim \frac{1}{2(s-1)} \frac{\|\phi\|^2}{\mathrm{Vol}(\Gamma\backslash\mathbb{H})},$$

and similarly for ψ. Therefore

$$I^+(s) \sim \frac{1}{2(s-1)} \frac{\mu_k^2 C_{-k-2}(1)\|\phi\|^2 - C_{-k}(1)\|\psi\|^2}{\Delta_k(1)\,\mathrm{Vol}(\Gamma\backslash\mathbb{H})}.$$

By [DFI02, eq. (4.3)], we have $\|\psi\|^2 = \mu_k \|\phi\|^2$, and thus

$$I_s(0) = I^+(s) \sim \frac{1}{2(s-1)} \frac{\mu_k(\mu_k C_{-k-2}(1) - C_{-k}(1))}{\Delta_k(1)} \frac{\|\phi\|^2}{\mathrm{Vol}(\Gamma\backslash\mathbb{H})}.$$

Finally, we compute

$$\frac{\mu_k(\mu_k C_{-k-2}(1) - C_{-k}(1))}{\Delta_k(1)} = \begin{cases} \sin(\pi s_\phi)/\pi, & (k=0) \\ \sin(\pi(s_\phi - \frac{1}{2}))/(\pi(s_\phi - \frac{1}{2})), & (k=1) \end{cases}$$

which agrees with $(\Gamma(s_\phi + \frac{k}{2})\Gamma(1 - s_\phi + \frac{k}{2}))^{-1}$ in both cases. This shows (8.25) with the claimed value of σ_ϕ^2.

We next upper-bound $I_s(h)$ for $h \neq 0$. By conjugation we may assume $h > 0$. We start by considering

$$\widetilde{I}_s(h) := \sum_{\substack{m,n \in \mathbb{Z}_{\neq 0} \\ n-m=h}} a(m)\overline{a(n)} b_h(m,n),$$

where for $m, n \in \mathbb{Z}_{\neq 0}$, we let

$$b_h(m,n) = \int_0^\infty W_{\mathrm{sgn}(m)k/2, it_\phi}(4\pi|m|y)\overline{W_{\mathrm{sgn}(n)k/2, it_\phi}(4\pi|n|y)} \mathrm{e}^{-2\pi h y} y^{2(s-1)} \frac{\mathrm{d}y}{y}.$$

For $n - m = h$, we have

$$\begin{aligned} b_h(m,n) &= \frac{1}{|n|^{2(s-1)}} \int_0^\infty W_{\mathrm{sgn}(m)k/2, it_\phi}(4\pi|\tfrac{m}{n}|y)\overline{W_{\mathrm{sgn}(n)k/2, it_\phi}(4\pi y)} y^{2(s-1)} \mathrm{e}^{-2\pi h y/n} \frac{\mathrm{d}y}{y} \\ &= \Big(1 + O_h\Big(\frac{1}{|n|}\Big)\Big) \frac{1}{|n|^{2(s-1)}} \int_0^\infty W_{\mathrm{sgn}(m)k/2, it_\phi}(4\pi y)\overline{W_{\mathrm{sgn}(n)k/2, it_\phi}(4\pi y)} y^{2(s-1)} \frac{\mathrm{d}y}{y} \\ &= \Big(1 + O_h\Big(\frac{1}{|n|}\Big)\Big) \frac{1}{|mn|^{s-1}} \int_0^\infty W_{\mathrm{sgn}(m)k/2, it_\phi}(4\pi y)\overline{W_{\mathrm{sgn}(n)k/2, it_\phi}(4\pi y)} y^{2(s-1)} \frac{\mathrm{d}y}{y}. \end{aligned} \tag{8.31}$$

Here in the second line we used the expression [DFI02, eq. (4.24)] to bound the derivative of the Whittaker function. We deduce that $b_h(m,n) \ll_h 1$. Also, if $\mathrm{sgn}(m) = \mathrm{sgn}(n) =$

± 1 by the definition 8.27 the integral in (8.31) can be expressed in terms of $C_{\pm k}$ giving

$$b_h(m,n) = \Big(1 + O_h\Big(\frac{1}{|n|}\Big)\Big)\frac{1}{|mn|^{s-1}}(4\pi)^{2(s-1)}C_{\pm k}(s). \tag{8.32}$$

Since there are finitely many indices m, n with $\operatorname{sgn}(m) \neq \operatorname{sgn}(n)$ (depending on h), the contribution to $\widetilde{I}_s(h)$ of these terms is $O_h(1)$. Thus, using (8.32) for the remaining terms, we obtain

$$\widetilde{I}_s(h) = (4\pi)^{2(s-1)}(C_k(s)I_h^+(s) + C_{-k}(s)I_h^-(s)) + O_h(1), \tag{8.33}$$

with

$$I_s^{\pm}(h) := \sum_{\substack{m,n\in\mathbb{Z}\\ \operatorname{sgn}(m)=\operatorname{sgn}(n)=\pm1\\ n-m=h}} a(m)\overline{a(n)}|mn|^{1-s}$$

since the contribution of the $O_h(1/n)$ is bounded by a constant times

$$\sum_{n\geq1}\frac{|a(n)|}{n}\sum_{n-m=h}|a(m)| \ll_h \sum_{n\geq1}\frac{|a(n)|}{n^{1-\varepsilon}} \ll 1$$

by the trivial bound $|a(m)| \ll m^{\varepsilon}$ and (2.5).

We can also give a different expression for $\widetilde{I}_s(h)$ by inserting the definition of b_h and exchanging the order of summation and integration. We obtain

$$\begin{aligned}\widetilde{I}_s(h) &= \int_0^\infty \sum_{m,n} a(m)\overline{a(n)}W_{\operatorname{sgn}(m)k/2,it_\phi}(4\pi my)\overline{W_{\operatorname{sgn}(n)k/2,it_\phi}(4\pi ny)}\mathrm{e}^{-2\pi hy}y^{2(s-1)}\frac{\mathrm{d}y}{y}\\ &= \int_{\Gamma\backslash\mathbb{H}} P_h(z,2s-1)|\phi(z)|^2\frac{\mathrm{d}x\,\mathrm{d}y}{y^2},\end{aligned}$$

where $P_h(z,2s-1)$ is the Poincaré series

$$P_h(z,2s-1) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}(\mathfrak{Im}\,\gamma z)^{2s-1}\mathrm{e}(h\gamma z).$$

It is known [Sel65, GS83] that $s \mapsto P_h(z,2s-1)$ for $h \neq 0$ extends to an analytic function in a neighborhood of $\{\mathfrak{Re}(s) \geq 1\}$ and satisfies $\int_{\Gamma\backslash\mathbb{H}}|P_h(z,2s-1)|^2\,\mathrm{d}x\,\mathrm{d}y/y^2 \ll_h 1$ uniformly for $s \geq 1$. Since $\phi(z) \ll 1$, we deduce by Cauchy–Schwarz that

$$\widetilde{I}_s(h) = \int_{\Gamma\backslash\mathbb{H}} P_h(z,2s-1)|\phi(z)|^2\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \ll_h 1,$$

so that, by (8.33),

$$C_k(s)I_h^+(s) + C_{-k}(s)I_h^-(s) = O_h(1)$$

as $s \to 1^+$. We carry out a similar argument for the level-raised form $\psi = R_k\psi$, and deduce by Cramer's rule as above that

$$I_h^+(s) = O_h(1), \qquad (s \to 1^+).$$

This proves (8.26), and completes the proof of (8.21).

A similar argument allows us to estimate $\int_{\mathbb{R}} L(\phi, s-1/2, x)^2\zeta_s(x)\,\mathrm{d}x$. The analogue of (8.23) is now

$$\int_{\mathbb{R}} L(\phi,s-1/2,x)^2\zeta_s(x)\,\mathrm{d}x \ll H^{-1}I_s(0) + \sum_{0<|h|\leq H}(|I_s(h)| + |J_s(h)|),$$

where

$$J_s(h) := \sum_{\substack{m,n\geq 1 \\ m+n=h}} \frac{a(m)a(n)}{(mn)^{2(s-1)}}.$$

In this sum, since m, n have the same sign, they are obviously bounded in terms of h, so that $J_s(h) \ll_h 1$ for all h. Letting $H \to \infty$ slowly enough as $s \to 1^+$ gives (8.22) as claimed, and concludes the proof of Proposition 8.6. □

9. Topological irreducibility with multiplier systems

We recall our assumptions on Γ, which follow from our setting in Section 2.1 as well as Lemma 5.1: Γ is discrete, finitely generated with finitely many cusps; it has a cusp at 1, and has rank at least 6.

9.1. **Topological irreducibility.** Recall that ϑ_g was defined in Section 5.1, and recall also the choice of fundamental domain made in Section 6.

We quote from [BDLb, eq. (3.1)] the notion of *K-length* ℓ_K. If $w = a_1 \cdots a_\ell \in \Sigma^*$ with $\ell \geq 1$, then

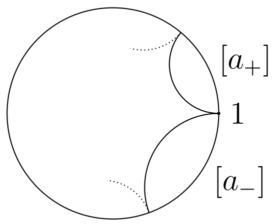
$$\ell_K(w) := 1 + |\{1 \leq i \leq \ell - 1,\ a_{i+1} - \widehat{a}_i \neq \pm 1\}|.$$

The goal of this section is to prove the following proposition, which shows that the multiplier system $\vartheta_g(x)$ satisfies the stability and mixing hypothesis from [BDLb, Section 5.1], or in other words, that $\vartheta_g(x)$ is irreducible with respect to the induced map $\mathcal{F}_K$ introduced therein. We shall use the fact that the fundamental domain R has at least 8 sides, as implied by Lemma 5.1.

Proposition 9.1. *Suppose $|\Sigma| \geq 8$. For any admissible non-empty words $w_1, w_3 \in \Sigma^*$, and any $\vartheta_0 \in \Theta$, there exists $\ell_0 \geq 1$ such that for all $\ell \geq \ell_0$, we may find $w_2 \in \Sigma^*$ with the following properties:*

- *The word $w_1 w_2 w_3$ is admissible, and $\ell_K(w_1 w_2 w_3) = \ell$.*
- *We have $\vartheta_{g_{w_1 w_2}}(x) = \vartheta_0$ for all $x \in [w_3]^\circ$.*

We first need some lemmas. In the following we denote by a_+ the side of R which precedes 1 proceeding counter-clockwise, and a_- the side which follows 1.



Lemma 9.2. *Let $w = a_1 \cdots a_\ell \in \Sigma^*$ with $\ell \geq 1$. Then the map*

$$x \mapsto \vartheta_{g_w}(x)$$

is constant on each interval $[b]^\circ$, for $b \neq \widehat{a_\ell}$. Denote by

$$\vartheta_w(b) = \vartheta_{g_w}(x), \qquad (x \in [b]^\circ)$$

the associated common value. The following properties hold.

(i) *The quantity $\vartheta_{\widehat{a_-}}(b)$ is independent of b for $b \neq a_-$.*
(ii) *Whenever $a_1, \widehat{a_\ell} \neq a_+$ and $g_w \cdot 1 \neq 1$, we have $\vartheta_w(a_+) = (-1)^{-k} \vartheta_{\widehat{w}}(a_+)^{-1}$.*
(iii) *For any $\eta \in \Theta$, we may find $w \in \Sigma^*$ such that $\vartheta_w(b) \in \{\pm\eta\}$ for any $b \neq \widehat{a_\ell}$.*
(iv) *Whenever $a_\ell \neq \widehat{a_1}$, we can find $n \geq 1$ such that $\vartheta_{w^n}(b) \in \{\pm 1\}$ for $b \neq \widehat{a_\ell}$.*

Proof. First we observe that if g_w fixes 1 (i.e. $[g_w]_{\mathbb{H}}$ fixes ∞), then $\vartheta_{g_w}(x) = d_w^k \chi'(g_w)$ for some $d_w \in \{\pm 1\}$ and all the claimed results follow immediately. Thus we can assume $g_w \cdot 1 \neq 1$. In particular, we can write ϑ_{g_w} as $\vartheta_{g_w} = c_w^{-k}\chi'(g_w)\,\mathrm{sg}_w(x)^k$ for some $c_w \in \{\pm 1\}$ satisfying $c_w = -c_{\widehat{w}}$ and where $\mathrm{sg}_w(x) = \mathrm{sgn}([x]_{\mathbb{H}} - [g_w]_{\mathbb{H}}^{-1}\infty)$. Notice that $\mathrm{sg}_w(x) = 1$ if $g_w^{-1} \cdot 1$ precedes x proceeding counter-clockwise from 1 and $\mathrm{sg}_w(x) = -1$ otherwise.

Since 1 is a vertex of R, and thus is not in $[a_1]^\circ$, we have $g_w^{-1} \cdot 1 \in [\widehat{a}_\ell]$. Thus for $x \in [b]^\circ$ and $b \neq \widehat{a}_\ell$ we have that $\mathrm{sg}_w(x)$ depends only on b and not on x. This justifies the definition of $\vartheta_w(b)$.

(i) If $g_{\widehat{a_-}}^{-1} \cdot 1 = 1$, then $g_{\widehat{a_-}}$ fixes 1 and the claim is clear as above. Now, assume $g_{\widehat{a_-}}^{-1} \cdot 1 \neq 1$. Since 1 is a vertex of R, we have $g_{\widehat{a_-}}^{-1} \cdot 1 = g_{a_-} \cdot 1 \in [a_-]$. This implies that $s_{\widehat{a_-}}(x) = 1$ for all $x \notin [a_-]$ and the claim follows.

(ii) We have $g_w^{-1} \cdot 1 \in [\widehat{a}_\ell]$ and thus, by hypothesis, $g_w^{-1} \cdot 1 \notin [a_+]^\circ$ so that $s_w(x) = 1$ for all $x \in [a_+]^\circ$. In the same way, $s_{\widehat{w}}(x) = 1$ for all $x \in [a_+]^\circ$ and the result follows since $c_w = -c_{\widehat{w}}$.

(iii) By definition of Θ, there exists $g \in G$ such that $\chi'(g) \in \{\pm\eta\}$. Write $g = g_w$ for some admissible word $w \in \Sigma^*$. Then $\vartheta_w(b) \in \{\pm\chi'(g)\} = \{\pm\eta\}$ as claimed.

(iv) We have $\vartheta_{w^n}(b) \in \{\pm\chi'(g_w)^n\}$. Since χ' takes values in the roots of unity of some fixed order, it suffices to pick n to be the common order of these roots. □

Given $w_1 \in \Sigma^*$ we extend the definition of ϑ_{w_1} also to words $w_2 \in \Sigma^*$, with $w_1 w_2$ admissible, by letting $\vartheta_{w_1}(w_2) := \vartheta_{w_1}(a)$ where a is the first letter of w_2.

We need some further notation. For any $s_1, s_2 \in \Sigma$ with s_1 adjacent to $\widehat{s_2}$, we let $P_{s_1,s_2} \in \Sigma^*$ be a (fixed) cuspidal cycle – as defined in (6.1) – with first letter s_1 and last letter s_2. Upon replacing P_{s_1,s_2} by a concatenation power P_{s_1,s_2}^r if necessary, we can impose that $\chi'(g_{P_{s_1,s_2}}) = 1$.

For $a_1, a_2 \in \Sigma$, we denote

$$E(a_1,a_2) = \left\{ b \in \Sigma, \begin{matrix} \widehat{a_1} \text{ not equal nor adjacent to } b \\ \text{and } \widehat{b} \text{ not equal nor adjacent to } a_2. \end{matrix} \right\}.$$

Note that since we assumed $|\Sigma| \geq 8$ we have $|E(a_1,a_2)| \geq 8 - 3 - 3 \geq 2$. In particular, $E(a_1,a_2)$ is never empty.

In the rest of the section we will repeatedly make use of the following simple facts about the function ℓ_K: let w_1 and w_2 be two non-empty admissible words. Assume that the last letter c_1 of w_1, and the first letter c_2 of w_2, are such that $\widehat{c_1}$ and c_2 are neither equal nor contiguous. Then

$$\ell_K(w_1 w_2) = \ell_K(w_1) + \ell_K(w_2). \tag{9.1}$$

Also, for s_1, s_2 as above we have $\ell_K(P_{s_1,s_2}) = 1$.

Lemma 9.3. *For any $w_3 \in \Sigma^*$, there exists $b \in \Sigma$ and $\vartheta_0 \in \Theta$ such that for any choice of $\eta \in \{(\pm 1)^{-k}\}$, we may find $u_2 \in \Sigma^*$ with bu_2w_3 admissible and*

$$\vartheta_{bu_2}(w_3) = \eta\vartheta_0, \qquad \ell_K(bu_2w_3) = \ell_K(w_3) + 4. \tag{9.2}$$

Proof. We pick $u \in \Sigma$ such that u does not belong to any of the contiguous intervals

$$\{a_+ - 1, a_+, a_-, a_- + 1, a_- + 2\}.$$

Since we assumed $|\Sigma| \geq 8 > 5$, there is at least one such choice of u. Let $v = \widehat{u+1}$, so that there is a cuspidal cycle $P_{u,v}$. Let

$$P \in \{P_{u,v}, \widehat{P_{u,v}}\},$$

and, given any $b' \in E(a_+, c)$ with c denoting the first letter of w_3, let

$$b := \widehat{a_-}, \qquad u_2 = Pa_+b' \in \Sigma^*.$$

By the period relation (5.6) we have

$$\vartheta_{bu_2}(w_3) = \vartheta_{\widehat{a_-}}(P)\vartheta_P(a_+)\vartheta_{a_+}(b') = \vartheta_0\vartheta_P(a_+)$$

where by Lemma 9.2.(i) the number $\vartheta_0 \in \Theta$ is independent of our choice of P. By Lemma 9.2.(ii), we have

$$\vartheta_P(a_+) = (-1)^{-k}\vartheta_{\widehat{P}}(a_+),$$

thus picking appropriately P, we find a corresponding word u_2 with $\vartheta_{bu_2}(w_3) = \eta\vartheta_0$. Moreover, a quick computation using the hypothesis on u and b' shows that

$$\ell_K(bu_2w_3) = \ell_K(\widehat{a_-}Pa_+b'w_3) = 4 + \ell_K(w_3).$$

This proves (9.2). □

Lemma 9.4. *For any $w_1 \in \Sigma^*$, any letter $b \in \Sigma$, and any $\vartheta_0 \in \Theta$, there exists $\ell_0' \in \mathbb{N}$ such that for any $\ell' \geq \ell_0'$, we may find a word $u_1 \in \Sigma^*$ with w_1u_1b admissible and*

$$\vartheta_{w_1u_1}(b) \in \{(\pm 1)^k\vartheta_0\}, \qquad \ell_K(w_1u_1b) = \ell'.$$

Proof. By the definition of Θ there exists $g \in G$ such that $\chi'(g_{w_1}g) \in \{(\pm 1)^k\vartheta_0\}$. Since G is spanned by $\{g_d, d \in \Sigma\}$ (see *e.g.* [Ser78, p. 79]), we may write $g = g_{d_1}\cdots g_{d_m}$ for some non-negative integer m and $d_j \in \Sigma$. Simplifying if necessary, we may assume $d_i \neq \widehat{d_j}$ for all $i \neq j$. Then, denoting by b_0 the last letter of w_1, we define $T \in \Sigma^*$ as follows:

— If $m = 0$ or if there exists j with $d_j \neq \widehat{b_0}$, in which case by reordering we may assume $j = 1$, we let $T = d_1 \cdots d_m$.
— Otherwise we have $m \geq 1$ and $g = \widehat{b_0}^m$. We then pick $e \in \Sigma \smallsetminus \{b_0, \widehat{b_0}\}$ and let $T = e\widehat{b_0}^m\widehat{e}$.

Note that T is the empty word if $m = 0$. In any case, the word w_1T is admissible and $\chi'(g_{w_1}g_T) \in \{(\pm 1)^k\vartheta_0\}$. We let $\ell_0' := \ell_K(w_1T) + 2$, and consider $\ell' \geq \ell_0'$ arbitrary. We let $r := \ell' - \ell_K(w_1T) - 1 \geq 1$ and we pick $a_1, \dots a_r \in \Sigma$ subject to the following constraints:

— a_1 is not equal nor adjacent to $\widehat{c}$, where c is the last letter of w_1T,
— for all $j \in \{2, \dots, r-1\}$, u_j is not equal nor adjacent to $a_{j-1} + 1$,
— $u_r \in E(\widehat{a_{r-1} + 1}, b)$.

For each j, we consider the cuspidal cycles $P_j := P_{a_j, \widehat{a_j+1}}$. We let

$$u_1 = TP_1 \cdots P_r \in \Sigma^*.$$

By construction, we have $\chi'(g_{w_1u_1}) \in \{(\pm 1)^k\vartheta_0\}$, and therefore

$$\vartheta_{w_1u_1}(b) \in \{(\pm 1)^k\vartheta_0\}.$$

Moreover, we have

$$\ell_K(w_1u_1b) = \ell_K(w_1TP_1 \cdots P_rb) = \ell_K(w_1T) + r + 1 = \ell',$$

as claimed. □

Proof of Proposition 9.1. Let w_1, w_3 be two given admissible words. We apply Lemma 9.3 for w_3, and let b and ϑ_1 be the quantities whose existence is asserted. We then apply Lemma 9.4 for the triple $(w_1, b, \vartheta_0\vartheta_1^{-1})$. Note that ℓ_0' depends on w_1, b and $\vartheta_0\vartheta_1^{-1}$. Since Σ and Θ are finite, we can pick $\ell_0 = \ell_0(w_3)$ large enough so that $\ell_0 \geq \ell_0' + \ell_K(w_3) + 3$.

Then, Lemma 9.4 applied to $\ell' = \ell - \ell_K(w_3) - 3 \geq \ell'_0$ yields the existence of a word $u_1 \in \Sigma^*$ and a sign $\eta \in \{(\pm 1)^k\}$ such that

$$\vartheta_{w_1 u_1}(b) = \eta \vartheta_0 \vartheta_1^{-1}, \qquad \ell_K(w_1 u_1 b) = \ell - \ell_K(w_3) - 3.$$

Since b and ϑ_1 are as given in Lemma 9.3, by the conclusion of the lemma we may find $u_2 \in \Sigma^*$ for which

$$\vartheta_{bu_2}(w_3) = \eta^{-1}\vartheta_1, \qquad \ell_K(bu_2 w_3) = \ell_K(w_3) + 4.$$

We let $w_2 = u_1 b u_2$. This word is admissible since $u_1 b$ and bu_2 both are. We readily get

$$\vartheta_{w_1 w_2}(w_3) = \vartheta_{w_1 u_1}(b)\vartheta_{bu_2}(w_3) = \vartheta_0,$$

and

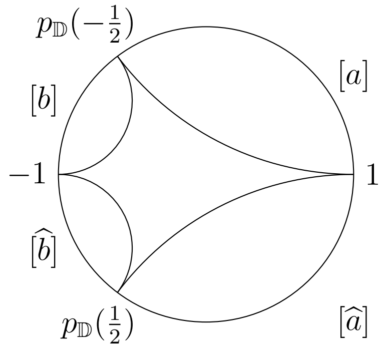
$$\ell_K(w_1 w_2 w_3) = \ell_K(w_1 u_1 b) + \ell_K(bu_2 w_3) - 1 = \ell,$$

as required. □

Remark 9.5. The condition $|\Sigma| \geq 8$ in Proposition 9.1 cannot be completely removed, as the following example shows: consider the group $\Gamma = \Gamma_0(4)$ with character $\chi(\gamma) := \chi_4(d)$, where $\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$ and χ_4 is the non-trivial character modulo 4. Then, for $x \in \mathbb{R}$

$$u_\gamma(x) = \operatorname{sgn}(cx + d)\chi_4(d), \tag{9.3}$$

Switching to the disk model as in Section 5 we may choose the following fundamental domain for $[\Gamma]_{\mathbb{D}} \backslash \mathbb{D}$:



It is easily seen that for any non-trivial word $w = a_1 \cdots a_n \in \Sigma^*$, $\ell_K(w)$ is the number of blocks in an alternating decomposition of w into subwords from $\{a, \widehat{b}\}^*$ and $\{\widehat{a}, b\}^*$. This implies, for instance, that any word $u \in \Sigma^*$ with aua admissible must have $\ell_K(aua)$ odd, so the conclusion of Proposition 9.1 does not hold for $w_1 = w_3 = a$ and ℓ even. Numerically, it seems that there is no such objection when $|\Sigma| = 6$, which leaves open the question as to whether $|\Sigma| \geq 6$ is a sufficient condition in Proposition 9.1.

Remark 9.6. We have also observed numerically that the conclusion of Proposition 9.1 holds for the first few values of ℓ when $\Gamma = \Gamma_0(4)$ with the multiplier system (9.3) granted that we replace ℓ_K by ℓ_c, defined in [BDLb, eq. (3.5)]. This provides evidence that the cuspidal acceleration $\mathcal{F}_c$ defined in [BDLb, Section 3.2] is topologically mixing in this case.

10. Theta series

In this section we explain the changes needed to the above argument to prove Theorem 1.9.

Let χ be a primitive Dirichlet character χ modulo N and let $\nu \in \{0,1\}$ be such that $\chi(-1)=(-1)^\nu$. Let θ_χ be defined by

$$\theta_\chi(z)=y^{(1+2\nu)/4}\sum_{n\in\mathbb{Z}}\chi(n)\mathrm{e}(n^2z),\qquad z\in\mathbb{H}. \tag{10.1}$$

For each $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0(4N^2)$ and $z\in\mathbb{H}$ we let

$$u_\gamma(z)=\varepsilon_d^{-1}\Big(\frac{c}{d}\Big)\chi(\gamma)j_\gamma(z)^{1/2+\nu}, \tag{10.2}$$

where $\chi(\gamma):=\chi(d)$, $(\frac{c}{d})$ is the Kronecker symbol,

$$\varepsilon_d=\Big(\frac{-1}{d}\Big)^{1/2}=\begin{cases}1 & d\equiv 1 \pmod 4,\\ i & d\equiv -1 \pmod 4,\end{cases}$$

and where the square root takes its principal value, i.e. its argument is in $(-\pi/2,\pi/2]$. Then, by [Shi73, Proposition 2.2] we have that θ_χ is a Maass forms of weight $k=\frac12+\nu$ for $\Gamma_0(4N^2)$, that is

$$\theta_\chi(\gamma z)=u_\gamma(z)\theta_\chi(z)\qquad\forall\gamma\in\Gamma_0(4N^2). \tag{10.3}$$

Also, by [SS77, p. 36] and [Iwa97, (10.54) and Corollary 10.7] we have that if χ is not totally even, then θ_χ is cuspidal, whereas if χ is totally even, then the constant term in the Fourier expansion of θ_χ at a cusp $\mathfrak{b}$ is of the shape $\theta_{\chi,\mathfrak{b}}(y)=A_{\mathfrak{b}}y^{1/4}$ (and in particular $B_{\mathfrak{b}}=0$ at all cusps). Thus, we set

$$\rho_{\theta_\chi}:=\rho_\chi=\begin{cases}1/4 & \chi\text{ totally even},\\ 0 & \text{otherwise},\end{cases}$$

as well as $s_\phi=1/4$.

We also need to extend the definition of u_γ to real arguments. Given $x\in\mathbb{R}\smallsetminus\{\gamma^{-1}\infty\}$ and $\gamma\notin\Gamma_\infty$, we let

$$u_\gamma(x):=\varepsilon_d^{-1}\Big(\frac{c}{d}\Big)\chi(\gamma)\,\mathrm{sgn}(c)^{-1/2-\nu}\Big(\frac{x-\gamma^{-1}\infty}{|x-\gamma^{-1}\infty|}\Big)^{1/2+\nu}=\lim_{y\to0^+}u_\gamma(x+iy). \tag{10.4}$$

Notice that if $c<0$ and $x>0$ this differs by a minus sign from what one obtains by simply setting $y=0$, but ensures that the cocycle condition still holds at real arguments.

The bounds (3.3)-(3.4) and [DN26, (2.11)][8] holds also for $\phi=\theta_\chi$, as the same proof still work in this case. In particular, we can replicate the arguments in Section 3[9] obtaining the analogous of Lemmas 3.2, 3.3, 3.5, 3.6 and 3.9 for θ_χ with the only changes that

— $\mathcal{D}_{\theta_\chi}:=\mathbb{C}\smallsetminus\{1/4,3/4\}$ for χ totally even and otherwise $\mathcal{D}_{\theta_\chi}=\mathbb{C}$.

— The last (harmless) error term in (3.15) is to be multiplied by $|cj(v,\alpha)|^{\rho_\phi}$.

As a consequence, Propositions 3.3 and 3.7 hold with the caveat that $h_\gamma^{\mathcal{E}}$ has a possible pole at $s=\frac12-s_\phi=\frac14$ if χ is totally even (in particular, in this case we need to add the condition $s-1/4\gg1$ in the bounds (3.12) and (3.19)) and that the asymptotics (3.13) and (3.20) needs to be slightly adapted in a straight-forward way. In any case the bounds (3.12) and (3.19) suffice for our purpose. The proof is again essentially identical with very minor changes. We signal in particular that

[8]As usual $y^{-1/2-\varepsilon}$ is to be replaced by $y^{-\rho_{\theta_\chi}-\varepsilon}$.

[9]We took care throughout Section 3 to chose the sign in the exponent in expressions such as $(-1)^{\pm k}$ in such a way that the claimed estimates hold true for $k\in\mathbb{Z}/2$ with the choice $(-1)^k=\mathrm{e}^{\pi ki}$.

— $\mathcal{M}_\beta(\phi, s)$ can still be absorbed in the bounds (3.12) and (3.19), since $B_{\mathfrak{b}} = 0$ at all cusps
— in the notation of Proposition 3.7, we have

$$\begin{aligned}[\phi^\sigma]^{\sigma^{-1}\gamma}(z) &= \Big(\lim_{y\to 0^+} j_{\sigma^{-1}\gamma}(x+it)^{-k} \cdot \lim_{y\to 0^+} j_\sigma(\sigma^{-1}\gamma(x+it))^k\Big) u_\gamma(x)\Phi(z) \\ &= c_{\sigma^{-1}\gamma}^{-k}(x-\omega)^k c_\sigma^{-k}(\sigma^{-1}\gamma x - \sigma^{-1}\infty)^k u_\gamma(x)\Phi(z).\end{aligned}$$

Finally, the identity (3.39) is replaced by the classical Mellin formula

$$\mathcal{E}_x(\theta_\chi, s) = (2\pi)^{-s+(1-2\nu)/4-}\Gamma(s-(1-2\nu)/4)L(\theta_\chi, s, x), \qquad \mathfrak{Re}(s) > 3/4 \tag{10.5}$$

where $L(\theta_\chi, s, x) = 2L_\chi(2s-1/2, x)$, with L_χ defined in (1.3).

The rest of the argument then follows essentially identically, with items (1), (2) and (4) of Lemma 7.1 holding unchanged, whereas (7.6) could either be slightly adapted or more simply replaced by the bound

$$\boldsymbol{\phi}_h(x) \ll n^{1/4} + |x-1|^{-1/4}$$

which ensures that $L(\theta_\chi, 1/2, x) = 2L_\chi(1/2, x)$ has moments of any order less than 4.

In order to obtain Theorem 1.9, it is then sufficient to verify that also the multiplier system corresponding to u_γ is irreducible with respect to the induced map $\mathcal{F}_K$, and to show that $\boldsymbol{\mu} = 0$ and $\boldsymbol{\Sigma} = \frac{\varphi(N)}{2N}\,\mathrm{Id}$.

Regarding the irreducibility, we use the smaller group $\Gamma(4N^2)$ to ensure R has at least 8 sides if $N = 1$. Then, we replicate the arguments of Section 9 as they are, with $\Theta = \{\pm i^a\chi(\gamma) \mid \gamma \in \Gamma(4N^2), a \in \{0,1\}\}$ and where, as usual, $(-1)^{\pm k}$ is to be interpreted as $\pm i(-1)^\nu$.

As for the mean and variance, one option would be to replicate the arguments of Section 8, however in this case we can also apply the approach of [BD22b, p.1416] and [DN26, Section 7.2], using standard analytic methods (the approximate functional equation for Dirichlet L-functions), which gives $\mathbb{E}_Q[L_\chi(1/2,x)] = o(\log Q)$, $\mathbb{E}_Q[L_\chi(1/2,x)^2] = o(\log Q)$ and $\mathbb{E}_Q[|L_\chi(1/2,x)|^2] \sim \frac{\varphi(N)}{N}\log Q$. In conjunction with the convergence of the third moment[10], this yield the claim.

Remark 10.1. If $f : \mathbb{Z} \to \mathbb{C}$ is N-periodic and odd or even, then Theorem 1.9 holds also for $L_f(s,x) := \sum_{n\ge 1} f(n)\frac{\mathrm{e}(n^2x)}{n^s}$ with variance $\sigma_f^2 := \frac{1}{2N}\sum_{n \pmod N}|f(n)|^2$. To see this, we express f in terms of primitive characters of conductor dividing N and deduce that for $z \in \mathbb{H}$

$$\theta_f(z) := y^{(1+2\nu_f)/4}\sum_{n\in\mathbb{Z}} f(n)\mathrm{e}(n^2z) = \sum_{d|N}\ \sideset{}{^*}\sum_{\substack{\chi \pmod d\\ \chi(-1)=(-1)^{\nu_f}}}\ \sum_{\ell|N/d}\alpha_{\chi,\ell}\theta_\chi(d^2z),$$

for some $\alpha_{\chi,\ell} \in \mathbb{C}$, where $\sum^*$ indicates the sum is restricted to primitive characters, and where $\nu_f = 0$ if f is even, and $\nu_f = 1$ if f is odd. Since $d\ell|N$, the old forms $\theta_\chi(d^2\,\cdot)$ are also Maass forms of weight $k = \frac12 + \nu_f$ for $\Gamma_0(4N^2)$. Although the multiplier system does depend on χ, this is bypassed by passing to the smaller subgroup $\Gamma_1(4N^2)$, where the characters become trivial. Consequently, $\theta_f(z)$ is a Maass form of weight k for $\Gamma_1(4N^2)$. By linearity we obtain the required properties and bounds for the period function associated to θ_f. Thus, the central limit theorem follows as above from the moment estimates $\mathbb{E}_Q[L_f(1/2,x)] = o(\sqrt{\log Q})$, $\mathbb{E}_Q[L_f(1/2,x)^2] = o(\log Q)$ and

[10]In [BD22b] the 4-th moment (9.6) is used. This is not available in this case when χ is totally even, but the argument still works using the 3-rd moment instead.

$\mathbb{E}_Q[|L_f(1/2,x)|^2] \sim 2\sigma_f(N)^2 \log Q$. The parity assumption on f seems superficial, and it would be nice to remove it. However, θ_χ has a different weight, and thus multiplier system, depending on wether χ is even or odd. A possible method to overcome this obstacle could be to define a variant of the operator $\mathscr{L}$, keeping track of the action of both systems. We won't pursue this here.

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