SMOOTH-SUPPORTED MULTIPLICATIVE FUNCTIONS IN ARITHMETIC PROGRESSIONS BEYOND THE $x^{1/2}$-BARRIER

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Abstract. We show that smooth-supported multiplicative functions f are well-distributed in arithmetic progressions $a_1a_2^{-1} \pmod q$ on average over moduli $q \le x^{3/5-\varepsilon}$ with $(q, a_1a_2) = 1$.

In memory of Klaus Roth

1. Introduction

In this paper we prove a Bombieri-Vinogradov type theorem for general multiplicative functions supported on smooth numbers, with a fixed member of the residue class. Given a multiplicative function f, we define, whenever $(a, q) = 1$,

$$\Delta(f, x; q, a) := \sum_{\substack{n\le x\\ n\equiv a \pmod q}} f(n) - \frac{1}{\varphi(q)} \sum_{\substack{n\le x\\ (n,q)=1}} f(n).$$

We wish to prove that, for an arbitrary fixed $A > 0$,

$$\sum_{\substack{q\sim Q\\ (a,q)=1}} |\Delta(f, x; q, a)| \ll \frac{x}{(\log x)^A} \tag{1.1}$$

where, here and henceforth, "$q \sim Q$" denotes the set of integers q in the range $Q < q \le 2Q$, for as large values of Q as possible. Let

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \quad \text{and} \quad -\frac{F'(s)}{F(s)} = \sum_{n=2}^{\infty} \frac{\Lambda_f(n)}{n^s},$$

for $\mathrm{Re}(s) > 1$. Following [5], we restrict attention to the class $\mathcal{C}$ of multiplicative functions f for which

$$|\Lambda_f(n)| \le \Lambda(n) \text{ for all } n \ge 1.$$

Date: April 18, 2017.
2010 *Mathematics Subject Classification.* 11N56.
Key words and phrases. Multiplicative functions, Bombieri-Vinogradov theorem, Siegel zeroes.
A.G. has received funding from the European Research Council grant agreement n° 670239, and from NSERC Canada under the CRC program.
X.S. was supported by a Glasstone Research Fellowship.

This includes most 1-bounded multiplicative functions of interest, including all 1-bounded completely multiplicative functions. Two key observations are that if $f \in \mathcal{C}$ then each $|f(n)| \le 1$, and if $f \in \mathcal{C}$ and $F(s)G(s) = 1$ then $g \in \mathcal{C}$.

In [6] the last two authors showed that there are two different reasons that the sum in (1.1) might be $\gg x/\log x$. First f might be a character of small conductor (for example $f(n) = (n/3)$), or might "correlate" with such a character; secondly f might have been selected so that $f(p)$ works against us for most primes p in the range $x/2 < p \le x$. We handled these potential pretentious problems as follows.

To avoid issues with the values $f(p)$ at the large primes p we only allow f to be supported on y-smooth integers[1] for $y = x^\theta$, for some small $\theta > 0$.

To avoid issues with the function f correlating with a given character χ, note that this happens when

$$S_f(X, \chi) := \sum_{n \le X} f(n)\overline{\chi}(n)$$

is "large" (that is, $\gg X$, or $\gg X/(\log X)^A$) for some X in the range $x^{1/2} < X \le x$, in which case (1.1) might well be false. We can either assume that this is false for all χ (which is equivalent to what is known as a "Siegel-Walfisz criterion" in the literature), or we can take account of such χ in the "Expected Main Term". We will begin by doing the latter, and then deduce the former as a corollary.

We start by stating the Siegel-Walfisz criterion:

The Siegel-Walfisz criterion: For any fixed $A > 0$, we say that f satisfies the A-Siegel-Walfisz criterion if for any $(a, q) = 1$ and any $x \ge 2$ we have the bound

$$|\Delta(f, x; q, a)| \ll_A \frac{1}{(\log x)^A} \sum_{n \le x} |f(n)|.$$

We say that f satisfies the Siegel-Walfisz criterion if it satisfies the A-Siegel-Walfisz criterion for all $A > 0$.

For a set of primitive characters Ξ, let Ξ_q be the set of those characters $\pmod q$ which are induced by the characters in Ξ. Then denote

$$\Delta_\Xi(f, x; q, a) := \sum_{\substack{n \le x \\ n \equiv a \pmod q}} f(n) - \frac{1}{\varphi(q)} \sum_{\chi \in \Xi_q} \chi(a) S_f(x, \chi)$$

In [6] we proved the following result:

Theorem 1.1. *Fix $\delta, B > 0$. Let $y = x^\varepsilon$ for some $\varepsilon > 0$ sufficiently small in terms of δ. Let $f \in \mathcal{C}$ be a multiplicative function which is only supported on y-smooth integers. Then there exists a set, Ξ, of primitive characters, containing $\ll (\log x)^{6B+7+o(1)}$ elements, such that for*

[1]That is, integers all of whose prime factors are $\le y$.

any $1 \le |a| \ll Q \le x^{\frac{20}{39}-\delta}$, *we have*

$$\sum_{\substack{q\sim Q\\(a,q)=1}} |\Delta_\Xi(f,x;q,a)| \ll \frac{x}{(\log x)^B}.$$

Moreover, if f *satisfies the Siegel-Walfisz criterion then*

$$\sum_{\substack{q\sim Q\\(a,q)=1}} |\Delta(f,x;q,a)| \ll \frac{x}{(\log x)^B}.$$

In this article we develop Theorem 1.1 further, allowing Q as well as y to vary over a much wider range, and obtaining upper bounds in terms of (the more appropriate) $\Psi(x,y)$, the number of y-smooth integers up to x.

Theorem 1.2. *Fix* $\varepsilon, A > 0$. *Suppose that* $f \in \mathcal{C}$, *and is only supported on* y*-smooth numbers, where*

$$x^\delta > y \ge \exp\left(\frac{5}{2}\cdot\frac{\sqrt{\log x}\log\log x}{\sqrt{\log\log\log x}}\right) \tag{1.2}$$

for some sufficiently small $\delta > 0$. *Then there exists a set,* Ξ, *of primitive characters, containing* $\ll (\log x)^{6A+38}$ *elements, such that if* $1 \le |a_1|, |a_2| \le x^\delta$ *then*

$$\sum_{\substack{q\le x^{3/5-\varepsilon}\\(q,a_1a_2)=1}} |\Delta_\Xi(f,x;q,a_1\overline{a_2})| \ll \frac{\Psi(x,y)}{(\log x)^A}.$$

Moreover, if f *satisfies the Siegel-Walfisz criterion then*

$$\sum_{\substack{q\le x^{3/5-\varepsilon}\\(q,a_1a_2)=1}} |\Delta(f,x;q,a_1\overline{a_2})| \ll \frac{\Psi(x,y)}{(\log x)^A}.$$

It would be interesting to extend the range (1.2) in Theorem 1.2 down to any $y \ge (\log x)^C$ for some large constant C. In our proofs we have used this range for y when we are able, as an aid to future research on this topic, and to make clear what are the sticking points.

Fouvry and Tenenbaum (Théorème 2 in [4]) established such a result when f is the characteristic function of the y-smooth integers (with $y < x^\delta$) and $a_2 = 1$, in the same range $q \le x^{3/5-\varepsilon}$, but with the bound $\ll x/(\log x)^A$. This was improved by Drappeau [2] to $\ll \Psi(x,y)/(\log x)^A$ for $(\log x)^C < y \le x^\delta$.

The proof of Theorem 1.2 combines the ideas from our earlier articles [2] and [6]. Perhaps the most innovative feature of this article, given [2] and [6], comes in Theorem 5.1 in which we prove a version of the classical large sieve inequality (towards which Roth's work [10] played a pivotal role) for (the notably sparse) sequences supported on the y-smooth numbers, which may be of independent interest.

2. Reduction to a larger set of exceptional moduli

We begin by modifying estimates from [2] to prove Theorem 2.1, which is a version of Theorem 1.2 with a far larger exceptional set of characters. This is key to the proof of Theorem 1.2 since we now only need to cope with relatively small moduli. We therefore define $\mathcal{A}(D)$ to be the set of all primitive characters of conductor $\le D$.

Theorem 2.1. *For fixed $\varepsilon, A > 0$, there exist $C, \delta > 0$ such that for any y in the range $(\log x)^C < y \le x^\delta$, and any $f \in \mathcal{C}$ which is only supported on y-smooth numbers, we have*

$$\sum_{\substack{q\le x^{3/5-\varepsilon}\\(q,a_1a_2)=1}} |\Delta_{\mathcal{A}}(f,x;q,a_1\overline{a_2})| \ll_A \frac{\Psi(x,y)}{(\log x)^A},$$

for any integers a_1, a_2 for which $1 \le |a_1|, |a_2| \le x^\delta$, with $\mathcal{A} = \mathcal{A}(D)$ where $D = (x/\Psi(x,y))^2(\log x)^{2A+20}$.

We prove this by modifying some of the estimates in [2]. For any $D \ge 1$ and integer $q \ge 1$ let

$$\mathfrak{u}_D(n;q) = \mathbf{1}_{n\equiv 1 \pmod q} - \frac{1}{\varphi(q)} \sum_{\substack{\chi \bmod q\\ \mathrm{cond}(\chi)\le D}} \chi(n), \tag{2.1}$$

so that

$$\Delta_{\mathcal{A}}(f,x;q,a_1\overline{a_2}) = \sum_{n\le x} f(n)\,\mathfrak{u}_D(n\overline{a_1}a_2;q).$$

Note that $\mathfrak{u}_D(n;q) = 0$ unless $(n,q) = 1$ and $q > D$, in which case

$$\begin{aligned} |\,\mathfrak{u}_D(n;q)| &\le \mathbf{1}_{n\equiv 1 \pmod q} + \frac{1}{\varphi(q)} \sum_{\substack{r\le D\\ r|q}} \varphi(r) \\ &\le \mathbf{1}_{n\equiv 1 \pmod q} + \frac{D\tau(q)}{\varphi(q)}. \end{aligned} \tag{2.2}$$

For $(n,q) = 1$, since

$$\sum_{\substack{\chi \pmod q\\ \mathrm{cond}(\chi)\le D}} \chi(n) = \sum_{\substack{s\le D\\ s|q}} \sum_{\substack{\psi \pmod s\\ \psi \text{ primitive}}} \psi(n) = \sum_{\substack{s\le D\\ s|q}} \sum_{d|s} \mu(s/d)\varphi(d)\mathbf{1}_{d|n-1},$$

by letting $b = s/d$ we obtain the alternate expression

$$\mathfrak{u}_D(n;q) = \mathbf{1}_{n\equiv 1 \pmod q} - \frac{1}{\varphi(q)} \sum_{\substack{d\le D\\ d|(q,n-1)}} \varphi(d) \sum_{\substack{b\le D/d\\ b|q/d}} \mu(b). \tag{2.3}$$

Theorem 2.1 is an immediate consequence of Theorem 2.2.

Theorem 2.2. *For any fixed $\varepsilon > 0$, there exists $C, \delta > 0$ such that whenever*

$$1 \le D \le x^{\delta}, \qquad (\log x)^C \le y \le x^{\delta},$$

we have, uniformly for $0 < |a_1|, |a_2| \le x^{\delta}$ and $f \in \mathcal{C}$,

$$\sum_{\substack{q \le x^{3/5-\varepsilon} \\ (q, a_1 a_2)=1}} \Big| \sum_{n \in S(x,y)} f(n)\, \mathfrak{u}_D(n\overline{a_1}a_2; q) \Big| \ll_{\varepsilon} D^{-\frac{1}{2}} x (\log x)^{10}. \tag{2.4}$$

To prove Theorem 2.2, we first prove the following generalisation of Theorem 3 of [2], where the bound D on the conductor is allowed to vary.

Lemma 2.3. *Let $M, N, L, R \ge 1$ and (α_m), (β_n), (λ_ℓ) be three sequences, bounded in modulus by 1, supported on integers inside $(M, 2M]$, $(N, 2N]$, and $(L, 2L]$ respectively. Let $x = MNL$. For any fixed $\varepsilon > 0$, there exists $\delta > 0$ such that whenever either the conditions (3.1), or the conditions (3.2) of* [2] *are met, we have*

$$\sum_{\substack{R < r \le 2R \\ (r, a_1 a_2)=1}} \Big| \sum_m \sum_n \sum_\ell \alpha_m \beta_n \lambda_\ell\, \mathfrak{u}_D(mn\ell\overline{a_1}a_2; q) \Big| \ll D^{-\frac{1}{2}} x (\log x)^3 \tag{2.5}$$

for $1 \le D \le x^{\delta}$.

Theorem 3 of [2] is the special case when D has maximal size, $D = x^{\delta}$. The conditions (3.1) or (3.2) of [2] concern the relative sizes of M, N, L. They are rather technical, but a critical case when the conditions are met is

$$R \approx x^{3/5}, \quad M \approx x^{1/5}, \quad N \approx x^{2/5}, \quad L \approx x^{2/5}.$$

Proof. We follow closely the arguments of [2]. Roughly speaking, the main point is that reducing the size of D only reduces the error terms, except in a certain diagonal contribution which yields the dominant error term, and which we analyse more carefully. Proceeding as in section 3 of [2], we reduce to the estimation of $\mathcal{S}_1 - 2\mathrm{Re}(\mathcal{S}_2) + \mathcal{S}_3$, where $\mathcal{S}_1$ is defined in the first display of [2, page 838],

$$\mathcal{S}_2 = \sum_{\substack{R < r \le 2R \\ (r, a_1 a_2)=1}} \frac{1}{\varphi(r)} \sum_{(m,r)=1} f(m) \sum_{\substack{(k_1, r)=1 \\ k_2 \equiv a_1 \overline{a_2 m} \bmod r}} \sum u_{k_1} \overline{u_{k_2}} \sum_{\substack{\chi \bmod r \\ \mathrm{cond}(\chi) \le D}} \chi(a_1 \overline{a_2 m k_2}),$$

and

$$\mathcal{S}_3 = \sum_{\substack{R < r \le 2R \\ (r, a_1 a_2)=1}} \frac{1}{\varphi(r)^2} \sum_{(m,r)=1} f(m) \sum_{(k_1 k_2, r)=1} \sum u_{k_1} \overline{u_{k_2}} \sum_{\substack{\chi_1, \chi_2 \bmod r \\ \mathrm{cond}(\chi_j) \le D}} \sum \chi_1 \overline{\chi_2}(a_1 \overline{a_2 m}) \chi_1(k_1) \overline{\chi_2(k_2)},$$

where $u_k = \sum_{k = n\ell} \beta_n \lambda_\ell$, and f is a smooth function supported inside $[M/2, 3M]$ satisfying $\|f^{(j)}\|_\infty \ll_j M^{-j}$ for any $j \ge 0$.

The quantity $\mathcal{S}_1$ being the same as in [2], we can quote the estimate

$$\mathcal{S}_1 = \hat{f}(0) X_1 + O(x^{1-\delta} K R^{-1})$$

from [2, formula (3.17)], where $K = NL$. Here $\hat{f}(0) = \int_{\mathbb{R}} f$, and X_1 is defined at [2, formula (3.12)]. For the estimation of $\mathcal{S}_2$ and $\mathcal{S}_3$, we reproduce sections 3.2 and 3.3 of [2], the only difference being that the set $\mathcal{X} = \{\chi \text{ primitive} : \text{ cond}(\chi) \le x^{\varepsilon}\}$ is replaced with the subset $\mathcal{X} = \{\chi \text{ primitive} : \text{ cond}(\chi) \le D\}$. We claim that the estimates

$$\mathcal{S}_j = \hat{f}(0)X_j + O(x^{1-\delta}KR^{-1})$$

hold for $j \in \{2, 3\}$, with

$$X_2 = \sum_{\substack{R<r\le 2R\\(r,a_1a_2)=1}} \frac{1}{r\varphi(r)} \sum\sum_{(k_1k_2,r)=1} \sum_{\substack{\chi \bmod r\\ \text{cond}(\chi)\le D}} u_{k_1}\overline{u_{k_2}}\chi(k_1\overline{k_2}),$$

$$X_3 = \sum_{\substack{R<r\le 2R\\(r,a_1a_2)=1}} \frac{1}{r} \sum_{\substack{0<b\le r\\(b,r)=1}} \left| \frac{1}{\varphi(r)} \sum_{(k,r)=1} \sum_{\substack{\chi \bmod r\\ \text{cond}(\chi)\le D}} u_k\chi(k\overline{b}) \right|^2.$$

To see this, we merely note that reducing the cardinality of $\mathcal{X}$, and the bound D on the conductors, leads to better error terms in the analysis. This is clear from the bound on R_3 in [2, formula (3.8)], which grows proportionally to $|\mathcal{X}|D^2$, and from the bound on $\sum_{z\in\mathcal{Z}} \lambda(z)|G_2(z,\xi)|$ in [2, formula (3.10)], which grows proportionally to $|\mathcal{X}|$.

Finally we are left with evaluating $X_1 - 2\text{Re}(X_2) + X_3$, which makes use of the multiplicative large sieve. Proceeding as in section 3.6 of [2], we find

$$\begin{aligned} &X_1 - 2\text{Re}(X_2) + X_3 \\ &\ll R^{-1}(\log R)^2 \sum_{d\le R} \frac{\tau(d)}{\varphi(d)} \int_D^{\infty} \Big(\min(2R,t)^2 + \frac{K}{d}\Big) \sum_{K/d<k'\le 2K/d} |u_{k'd}|^2 \frac{\mathrm{d}t}{t^2} \\ &\ll R^{-1}(\log R)^2 K(\log K)^3 \sum_{d\le R} \frac{\tau(d)^3}{d\varphi(d)} \Big(\frac{K}{dD} + R\Big) \\ &\ll (\log x)^5 K^2 (RD)^{-1}. \end{aligned} \tag{2.6}$$

Here we have used the bound[2] $|u_k| \le \tau(k)$, and the hypothesis $R \le x^{-\varepsilon}K \le K/D$. Following [2, formula (3.34)], this leads to the upper bound

$$\begin{aligned} &\sum_{\substack{R<r\le 2R\\(r,a_1a_2)=1}} \Big| \sum_m \sum_n \sum_\ell \alpha_m\beta_n\lambda_\ell\, \mathfrak{u}_D(mn\ell\overline{a_1}a_2; q) \Big| \\ &\ll (M^2R\{X_1 - 2\text{Re}(X_2) + X_3\})^{1/2} + O(x^{1-\delta/3}). \end{aligned}$$

The claimed bound (2.5) then follows by (2.6). □

To deduce Theorem 2.2 from Lemma 2.3, we start with the following special case of Theorem 2.2.

[2]Note that there is a factor $(\tau(d)\log K)^2$ missing in the third display, p.852 of [2].

Proposition 2.4. *Theorem 2.2 holds true for functions f supported on squarefree integers.*

Proof. We extend the arguments of pages 852-853 of [2], renaming the variable q into r. Suppose first $R \geq x^{4/9}$. We restrict n and r to dyadic intervals $x < n \leq 2x$ and $R < r \leq 2R$. Choosing the parameters (M_0, N_0, L_0) as in [2, p.852, last display], we obtain

$$\begin{aligned}
&\sum_{\substack{R<r\leq 2R\\(r,a_1a_2)=1}} \Big| \sum_{\substack{x<n\leq 2x\\P^+(n)\leq y}} f(n)\,\mathfrak{u}_D(n\overline{a_1}a_2;r)\Big| \\
&= \sum_{\substack{R<r\leq 2R\\(r,a_1a_2)=1}} \Big| \sum_{\substack{L_0<\ell\leq L_0P^-(\ell)\\P^+(\ell)\leq y}} \sum_{\substack{M_0<m\leq M_0P^-(m)\\P^+(m)\leq P^-(\ell)}} \sum_{\substack{x<mn\ell\leq 2x\\P^+(n)\leq P^-(m)}} f(mn\ell)\,\mathfrak{u}_D(mn\ell\overline{a_1}a_2;r)\Big| \\
&= \sum_{\substack{R<r\leq 2R\\(r,a_1a_2)=1}} \Big| \sum_{\substack{L_0<\ell\leq L_0P^-(\ell)\\P^+(\ell)\leq y}} \sum_{\substack{M_0<m\leq M_0P^-(m)\\P^+(m)< P^-(\ell)}} \sum_{\substack{x<mn\ell\leq 2x\\P^+(n)< P^-(m)}} f(m)f(n)f(\ell)\,\mathfrak{u}_D(mn\ell\overline{a_1}a_2;r)\Big|
\end{aligned}$$

where we have used the fact that f is supported on squarefree integers in the last equality. The rest of the argument consists in cutting the sums over (m, n, ℓ) in dyadic segments, and analytically separating the four conditions $x < mn\ell \leq 2x$, $P^+(m) < P^-(\ell)$ and $P^+(n) < P^-(m)$. The details are identical to the proof of Proposition 2 of [2], using our Lemma 2.3 instead of [2, Theorem 3]; we obtain the bound

$$O(D^{-\frac{1}{2}}x(\log x)^7(\log y)^3) = O(D^{-\frac{1}{2}}x(\log x)^{10}).$$

The Bombieri-Vinogradov range $R \leq x^{4/9}$ is covered by similar arguments, using [9, Theorem 17.4] instead of Lemma 2.3. □

Deduction of the full Theorem 2.2 from Proposition 2.4. We let $\mathcal{K}$ be the set of powerful numbers, that is for $k \in \mathcal{K}$ if prime p divides k then p^2 also divides k. Note that $|\mathcal{K} \cap [1, x]| \ll x^{\frac{1}{2}}$. Out of every n counted in the left-hand side of (2.4), we extract the largest powerful divisor k. Then from the triangle inequality and the bound $|f(k)| \leq 1$, the left-hand side of (2.4) is at most

$$\sum_{\substack{q\leq x^{3/5-\varepsilon}\\(q,a_1a_2)=1}} \sum_{\substack{k\in\mathcal{K}\cap S(x,y)\\(k,q)=1}} \Big| \sum_{\substack{n\in S(x/k,y)\\(n,k)=1}} \mu^2(n)f(n)\,\mathfrak{u}_D(kna_2\overline{a_1};q)\Big|. \tag{2.7}$$

Let $K \geq 1$ be a parameter. We use the trivial bound (2.2) on the contribution of $k > K$, getting

$$\sum_{q\leq x^{3/5-\varepsilon}} \sum_{\substack{k\in\mathcal{K}\\k>K}} \sum_{n\leq x/k} \Big(\mathbf{1}_{ka_2n\equiv a_1 \mod q} + \frac{D\tau(q)}{\varphi(q)}\Big) = T_1 + T_2,$$

say, where we have separated the contribution of the two summands. Executing the sum over q first, and separating the case $kn|a_1$, we find

$$T_1 \le \sum_{q\le x^{3/5-\varepsilon}} \tau(|a_1|)^2 + \sum_{\substack{k\in\mathcal{K}\\ k>K}} \sum_{n\le x/k} \tau(|kna_2 - a_1|) \ll x^{4/5} + x^{1+\varepsilon} DK^{-\frac12}.$$

It is easy to see that $T_2 \ll x^{1+\varepsilon}DK^{-\frac12}$ as well. Next, to each $1 \le k \le K$ in (2.7), by hypothesis, we may use Proposition 2.4 with $x \leftarrow x/k$, $a_2 \leftarrow ka_2$ and $f(n) \leftarrow \mathbf{1}_{(n,k)=1}\mu^2(n)f(n)$, and obtain the existence of $C, \delta_1 > 0$ such that, for $|a_1|, |a_2k| \le x^{\delta_1}$ and $(\log x)^C \le y \le x^{\delta_1}$,

$$\sum_{\substack{q\le x^{3/5-\varepsilon}\\ (q,a_1a_2)=1}} \Big| \sum_{\substack{n\in S(x/k,y)\\ (n,k)=1}} \mu^2(n)f(n)\,\mathfrak{u}_D(n\overline{a_1}ka_2;q)\Big| \ll D^{-\frac12}k^{-1}x(\log x)^{10}.$$

We take $\delta = \delta_1/7$, $K = x^{\delta_1/2}$, and sum over $k \le K$, using $\sum_{k\in\mathcal{K}} k^{-1} < \infty$. By hypothesis $D \le x^\delta$, so that $x^{4/5} \ll DK^{-\frac12}x \ll D^{-\frac12}x^{1-\delta/2}$, and we find that (2.7) is at most $\ll D^{-\frac12}x(\log x)^{10}$ as claimed. □

3. Altering the set of exceptional characters

To prove Theorem 1.2 we need to reduce the set of exceptional characters from $\mathcal{A}(D)$ to Ξ. We shall set this up in Proposition 3.2.

It is convenient to write $b = a_1/a_2$ (which is $\equiv a_1\overline{a_2} \pmod q$) and to define (q, b) to mean (q, a_1a_2). Thus in Theorem 2.1 we are working with

$$\sum_{\substack{q\le Q\\ (q,b)=1}} |\Delta_{\mathcal{A}}(f, x; q, b)|$$

for $Q = x^{3/5-\varepsilon}$.

Lemma 3.1. *Let $\mathcal{A} = \mathcal{A}(D)$ for some $D \ge 2$. Suppose that $\Xi \subset \mathcal{A}$. If $(b, q) = 1$ then*

$$\begin{aligned}&\Delta_\Xi(f, x; q, b) - \Delta_{\mathcal{A}}(f, x; q, b)\\ &= \frac{1}{\varphi(q)} \sum_{\substack{\ell\ge1\\ p|\ell \Longrightarrow p|q}} g(\ell) \sum_{\substack{d\le D\\ (d,\ell)=1\\ d|q}} \varphi(d)\Delta_\Xi(f, x/\ell; d, b\overline{\ell}) \sum_{\substack{n\le D/d\\ n|q/d}} \mu(n),\end{aligned}$$

where g is the multiplicative function with $F(s)G(s) = 1$.

Proof. If $(b,q)=1$ then

$$\Delta_\Xi(f,x;q,b)-\Delta_{\mathcal{A}}(f,x;q,b)=\frac{1}{\varphi(q)}\sum_{\substack{\chi\in\mathcal{A}_q\\ \chi\notin\Xi_q}}\chi(b)S_f(x,\chi)$$
$$=\frac{1}{\varphi(q)}\sum_{\substack{m\le D\\ m|q}}\sum_{\substack{\chi\in\mathcal{P}(m)_q\\ \chi\notin\Xi_q}}\chi(b)S_f(x,\chi),$$

where $\mathcal{P}(m)$ denotes the set of primitive characters $\pmod m$, as $\mathcal{A}$ is the set of all primitive characters of conductor $\le D$. Let $\mathcal{C}(m)$ denote the set of all characters $\pmod m$. For $m|q$ we define

$$\Delta_{\Xi,q}(f,x;m,b):=\sum_{\substack{n\le x\\ n\equiv a \pmod m\\ (n,q)=1}}f(n)-\frac{1}{\varphi(m)}\sum_{\chi\in\mathcal{C}(m)_q\cap\Xi_q}\chi(b)S_f(x,\chi)$$
$$=\frac{1}{\varphi(m)}\sum_{\substack{\chi\in\mathcal{C}(m)_q\\ \chi\notin\Xi_q}}\chi(b)S_f(x,\chi)$$
$$=\frac{1}{\varphi(m)}\sum_{d|m}\sum_{\substack{\chi\in\mathcal{P}(d)_q\\ \chi\notin\Xi_q}}\chi(b)S_f(x,\chi).$$

By Möbius inversion we deduce that, for $m|q$,

$$\sum_{\substack{\chi\in\mathcal{P}(m)_q\\ \chi\notin\Xi_q}}\chi(b)S_f(x,\chi)=\sum_{d|m}\mu(m/d)\varphi(d)\Delta_{\Xi,q}(f,x;d,b).$$

Next we wish to better understand $\Delta_{\Xi,q}(f,x;m,a)$. Let $f_q(p^k)=f(p^k)$ if $p|q,\ p\nmid m$, and $f_q(p^k)=0$ otherwise. Define g_q from g in a similarly way. Note f_q and g_q are simply f and g supported on the integers composed from the prime factors of q. If $(a,m)=1$ then

$$\Delta_\Xi(f,x;m,a)=\sum_{\substack{\ell\ge1\\ (\ell,m)=1}}f_q(\ell)\Delta_{\Xi,q}(f,x/\ell;m,a\overline{\ell}),$$

and since $F_qG_q=1$ we have

$$\Delta_{\Xi,q}(f,x;m,a)=\sum_{\substack{\ell\ge1\\ (\ell,m)=1}}g_q(\ell)\Delta_\Xi(f,x/\ell;m,a\overline{\ell})$$
$$=\sum_{\substack{\ell\ge1\\ p|\ell\Longrightarrow p|q\\ (\ell,m)=1}}g(\ell)\Delta_\Xi(f,x/\ell;m,a\overline{\ell}).$$

Substituting this in above then yields

$$\sum_{\substack{\chi\in\mathcal{P}(m)_q\\ \chi\notin\Xi_q}}\chi(b)S_f(x,\chi)=\sum_{d|m}\mu(m/d)\varphi(d)\sum_{\substack{\ell\geq 1\\ p|\ell\Longrightarrow p|q,\\ (\ell,d)=1}}g(\ell)\Delta_\Xi(f,x/\ell;d,b\overline{\ell}),$$

and the result follows writing $m=dn$. □

Proposition 3.2. *Let the notations and assumptions be as in the statement of Theorem 2.1. Suppose that $\Xi\subset\mathcal{A}$. Then*

$$\sum_{\substack{q\leq x^{3/5-\varepsilon}\\ (b,q)=1}}|\Delta_\Xi(f,x;q,b)|\leq O\left(\frac{\Psi(x,y)}{(\log x)^A}\right)$$
$$+(\log x)^2\sum_{\substack{\ell\leq X\\ L:=\prod_{p|\ell}p}}\frac{\tau(L)}{\varphi(L)}\sum_{\substack{d\leq D\\ (d,\ell)=1}}|\Delta_\Xi(f,x/\ell;d,b\overline{\ell})|,$$

where $X=(1+\beta)^2(\log x)^{2A+6}$ with

$$\beta=\beta(\Xi):=\sum_{\psi\pmod{r_\psi}\in\Xi}\frac{1}{r_\psi}.$$

Proof. Set $Q=x^{3/5-\varepsilon}$. We deduce from Lemma 3.1, as each $|g(\ell)|\leq 1$ since $f,g\in\mathcal{C}$, that

$$\sum_{\substack{q\leq Q\\ (b,q)=1}}|\Delta_\Xi(f,x;q,b)|\leq\sum_{\substack{q\leq Q\\ (b,q)=1}}|\Delta_\mathcal{A}(f,x;q,b)|$$
$$+\sum_{\substack{\ell\geq 1\\ L:=\prod_{p|\ell}p}}\sum_{\substack{d\leq D\\ (d,\ell)=1}}\varphi(d)|\Delta_\Xi(f,x/\ell;d,b\overline{\ell})|\sum_{\substack{q\leq Q\\ (b,q)=1\\ dL|q}}\frac{1}{\varphi(q)}\left|\sum_{\substack{n\leq D/d\\ n|q/d}}\mu(n)\right|.$$

The first term on the right-hand side is $\ll_A\Psi(x,y)/(\log x)^A$ by Theorem 2.1. For the sum at the end we have an upper bound

$$\leq\sum_{\substack{q\leq Q\\ dL|q}}\frac{\tau^*(q/d)}{\varphi(q)}\leq\frac{\tau^*(L)}{\varphi(dL)}\sum_{r\leq Q/dL}\frac{\tau^*(r)}{\varphi(r)}\ll\frac{\tau(L)}{\varphi(d)\varphi(L)}(\log x)^2,$$

where $\tau^*(m)$ denotes the number of squarefree divisors of m, writing $q = dLr$, as L is squarefree. Therefore

$$\sum_{\substack{q\le Q\\(b,q)=1}} |\Delta_\Xi(f,x;q,b)| \le O\left(\frac{\Psi(x,y)}{(\log x)^A}\right)$$
$$+ (\log x)^2 \sum_{\substack{\ell\ge 1\\ L:=\prod_{p\mid \ell} p}} \frac{\tau(L)}{\varphi(L)} \sum_{\substack{d\le D\\(d,\ell)=1}} |\Delta_\Xi(f,x/\ell;d,b\overline{\ell})|.$$

We will attack this last sum first by employing relatively trivial bounds for the terms with ℓ that are not too small, so that we only have to consider ℓd that are smallish in further detail. Now Theorem 1 of [3] gives the upper bound

$$\Psi_q(x,y) \ll \frac{\varphi(q)}{q}\Psi(x,y) \tag{3.1}$$

provided $x \ge y \ge \exp((\log\log x)^2)$ and $q \le x$. Therefore

$$|\Delta_\Xi(f,x;q,a)| \le \Psi(x,y;a,q) + \frac{|\Xi_q|}{\varphi(q)}\Psi_q(x,y) \ll \Psi(x,y;a,q) + \frac{|\Xi_q|}{q}\Psi(x,y)$$

in this range. Substituting in, the upper bound on the ℓth term above becomes

$$\ll (\log x)^2 \cdot \frac{\tau(L)}{\varphi(L)}\left(\sum_{\substack{d\le D\\(d,\ell)=1}} \Psi(x/\ell,y;d,b\overline{\ell}) + \Psi(x/\ell,y)\sum_{\substack{d\le D\\(d,\ell)=1}} \frac{|\Xi_d|}{d}\right).$$

The second sum over d is therefore

$$\sum_{\psi \pmod{r_\psi}\in\Xi} \sum_{\substack{d\le D\\(d,\ell)=1\\ r_\psi\mid d}} \frac{1}{d} \ll \beta\frac{\varphi(\ell)}{\ell}\log D.$$

For the first term we use Theorem 1 of [7] which yields that

$$\sum_{\substack{d\le D\\(d,\ell)=1}} \left|\Psi(x/\ell,y;d,b\overline{\ell}) - \frac{\Psi_d(x/\ell,y)}{\varphi(d)}\right| \ll \frac{\Psi(x/\ell,y)}{(\log x)^A},$$

as $D \le \sqrt{\Psi(x,y)}/(\log x)^B$ since $y \ge (\log x)^C$. Therefore, expanding the sum and using (3.1), we obtain

$$\sum_{\substack{d\le D\\(d,\ell)=1}} \Psi(x/\ell,y;d,b\overline{\ell}) \ll \sum_{\substack{d\le D\\(d,\ell)=1}} \frac{\Psi(x/\ell,y)}{d} + \frac{\Psi(x/\ell,y)}{(\log x)^A} \ll \frac{\varphi(\ell)}{\ell}\log D\cdot\Psi(x/\ell,y).$$

By Théorème 2.1 of [1] we have

$$\Psi(x/\ell, y) \ll \Psi(x,y)/\ell^\alpha, \tag{3.2}$$

where $\alpha > 3/4$ in our range for y. Therefore in total the ℓth term in

$$\ll \frac{\tau(L)}{L\ell^\alpha}\Psi(x,y)(\log x)^2(\log D)\cdot(1+\beta).$$

Summing over $\ell > X$, we obtain, taking $\sigma = \alpha - 1/4$,

$$\sum_{\substack{\ell\ge X\\ L:=\prod_{p|\ell}p}} \frac{\tau(L)}{L\ell^\alpha} \le \sum_{\substack{\ell\\ L:=\prod_{p|\ell}p}} \frac{\tau(L)}{L\ell^\alpha}(\ell/X)^\sigma = X^{-\sigma}\prod_p\left(1+\frac{2}{p(p^{1/4}-1)}\right) \ll X^{-1/2}.$$

Taking $X = (1+\beta)^2(\log x)^{2A+6}$, the contribution of the $\ell > X$ is therefore $\ll \Psi(x,y)/(\log x)^A$. Combining all of the above then yields the result. □

4. Putting the pieces together

In order to prove Theorem 1.2 we need Theorem 2.1, Proposition 3.2, Corollary 6.1 (which will be proved in the final two sections), and the following result which is Proposition 5.1 of [6].

Proposition 4.1. *Fix $B \ge 0$ and $0 < \eta < \frac12$. Given $(\log x)^{4B+5} \le y = x^{1/u} \le x^{1/2-\eta}$ let*

$$R = R(x,y) := \min\{y^{\frac{\log\log\log x}{3\log u}}, x^{\frac{\eta}{3\log\log x}}\}\ (\le y^{1/3}).$$

Suppose that $f \in \mathcal{C}$, and is only supported on y-smooth numbers. There exists a set, Ξ, of primitive characters $\psi \pmod r$ with $r \le R$, such that if $q \le R$ and $(a,q) = 1$ then

$$|\Delta_\Xi(f,x;q,a)| \ll \frac{1}{\varphi(q)}\frac{\Psi(x,y)}{(\log x)^B}.$$

Moreover, one may take Ξ to be $\Xi = \Xi(2B+2\frac13)$, where $\Xi(C)$ is the set of primitive characters $\psi \pmod r$ with $r \le R$ such that there exists $x^\eta < X \le x$ for which

$$|S_f(X,\psi)| \ge \frac{\Psi(X,y)}{(u\log u)^4(\log x)^C}. \tag{4.1}$$

Proof of Theorem 1.2. Let $c > 0$ be the small constant from Corollary 6.1. Set $D = (x/\Psi(x,y))^2(\log x)^{2A+20}$, and one easily verifies that the hypothesis for y implies that

$$D \le \min(R, y^c, \exp(c\log x/\log\log x))$$

from the usual estimate $\Psi(x,y) = xu^{-u+o(u)}$ for smooth numbers. We will prove Theorem 1.2 with $\Xi = \Xi(2A+8\frac13)$, where $\Xi(C)$ is the set of primitive characters $\psi \pmod r$ with $r \le D$, such that there exists $x^{1/4} < X \le x$ for which (4.1) holds. By Proposition 4.1 with $B = A+3$ and $\eta = 1/4$, we have the bound

$$|\Delta_\Xi(f,x;q,a)| \ll \frac{1}{\varphi(q)}\frac{\Psi(x,y)}{(\log x)^{A+3}}$$

whenever $q \le D$ and $(a,q)=1$. Moreover, we have the same bound with x replaced by x/ℓ for any $\ell = x^{o(1)}$.

The goal of the next two sections will be to prove Corollary 6.1, which implies that

$$|\Xi| \ll (\log x)^{6A+38}.$$

This implies that $\beta(\Xi) \ll (\log x)^{3A+19}$, and so $X \ll (\log x)^{8A+44}$ in Proposition 3.2. Thus for each $\ell \le X$ and $d \le D$, we have

$$|\Delta_\Xi(f, x/\ell; d, b\overline{\ell})| \ll \frac{1}{\varphi(d)} \frac{\Psi(x/\ell, y)}{(\log x)^{A+3}} \ll \frac{1}{\varphi(d)\ell^\alpha} \frac{\Psi(x,y)}{(\log x)^{A+3}},$$

by (3.2). Therefore

$$\begin{aligned}
&(\log x)^2 \sum_{\substack{\ell \le X \\ L := \prod_{p|\ell} p}} \frac{\tau(L)}{\varphi(L)} \sum_{\substack{d \le D \\ (d,\ell)=1}} |\Delta_\Xi(f, x/\ell; d, b\overline{\ell})| \\
&\ll \frac{\Psi(x,y)}{(\log x)^{A+1}} \sum_{\substack{\ell \le X \\ L := \prod_{p|\ell} p}} \frac{\tau(L)}{\varphi(L)\ell^\alpha} \sum_{\substack{d \le D \\ (d,L)=1}} \frac{1}{\varphi(d)} \ll \frac{\Psi(x,y)}{(\log x)^A} \sum_{\substack{\ell \le X \\ L := \prod_{p|\ell} p}} \frac{\tau(L)}{L\ell^\alpha} \\
&\le \frac{\Psi(x,y)}{(\log x)^A} \prod_{p \le X} \left(1 + \frac{2}{p(p^\alpha - 1)}\right) \ll \frac{\Psi(x,y)}{(\log x)^A}.
\end{aligned}$$

We therefore deduce from Proposition 3.2 that

$$\sum_{\substack{q \le x^{3/5-\varepsilon} \\ (q, a_1 a_2)=1}} |\Delta_\Xi(f, x; q, a_1\overline{a_2})| \ll \frac{\Psi(x,y)}{(\log x)^A}$$

as desired.

To deduce the second part of Theorem 1.2, about functions f satisfying the Siegel-Walfisz criterion, we use the following variant of Proposition 3.4 in [6]:

Proposition 4.2. *Fix $\varepsilon > 0$. Let $(\log x)^{1+\varepsilon} \le y \le x$ be large. Let $f \in \mathcal{C}$ be a multiplicative function supported on y-smooth integers. Suppose that Ξ is a set of primitive characters, containing $\ll (\log x)^C$ elements, such that*

$$\sum_{q \sim Q} |\Delta_\Xi(f, x; q, a_q)| \ll \frac{\Psi(x,y)}{(\log x)^B},$$

for $(a_q, q) = 1$ for all $q \sim Q$, where $Q \le x$. If the D-Siegel-Walfisz criterion holds for f, where $D \ge B + C$, then

$$\sum_{q \sim Q} |\Delta(f, x; q, a_q)| \ll \frac{\Psi(x,y)}{(\log x)^B}.$$

Proof. By the definition of Δ_Ξ we have

$$|\Delta(f,x;q,a_q)| \le |\Delta_\Xi(f,x;q,a_q)| + \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \chi\in\Xi_q, \chi\neq\chi_0}} |S_f(x,\chi)|.$$

Summing this over $q \sim Q$ and using the hypothesis, we deduce that

$$\sum_{q\sim Q} |\Delta(f,x;q,a_q)| \le \sum_{\substack{\psi\in\Xi \\ \psi\neq 1}} \sum_{\substack{r_\psi | q\sim Q \\ \chi \pmod q \text{ induced by } \psi}} \frac{|S_f(x,\chi)|}{\varphi(q)} + O\left(\frac{\Psi(x,y)}{(\log x)^B}\right).$$

It suffices to show that, for each fixed $\psi \pmod r \in \Xi$ with $r>1$, we have

$$\sum_{\substack{r|q\sim Q \\ \chi \pmod q \text{ induced by } \psi}} \frac{|S_f(x,\chi)|}{\varphi(q)} \ll \frac{\Psi(x,y)}{(\log x)^D}. \tag{4.2}$$

The conclusion then follows since $|\Xi| \ll (\log x)^C$ and $D \ge B+C$. If $\chi \pmod q$ is induced by $\psi \pmod r$, then there is a multiplicative function h supported only on powers of primes which divide q but not r, such that $h * f\overline{\psi} = f\overline{\chi}$. Note that $h \in \mathcal{C}$ since $f \in \mathcal{C}$, and in particular h is 1-bounded. It follows that

$$|S_f(x,\chi)| = \left|\sum_{m\le x} h(m) S_f(x/m,\psi)\right| \le \sum_{\substack{m\le x \\ p|m\Rightarrow p|q, p\nmid r}} |S_f(x/m,\psi)|.$$

Since f satisfies the D-Siegel-Walfisz criterion, we have

$$\frac{S_f(x/m,\psi)}{\varphi(r)} = \frac{1}{\varphi(r)} \sum_{a \pmod r} \overline{\psi}(a)\Delta(f,x/m;r,a) \ll \frac{\Psi(x/m,y)}{(\log(x/m))^D}.$$

Using the bound $\Psi(x/m,y) \ll m^{-\alpha}\Psi(x,y)$ where $\alpha = \alpha(x,y) \ge \varepsilon + o(1)$, we may bound the left hand side of (4.2) by

$$\Psi(x,y) \sum_{r|q\sim Q} \frac{\varphi(r)}{\varphi(q)} \sum_{\substack{m\le x \\ p|m\Rightarrow p|q, p\nmid r}} \frac{1}{m^\alpha (\log(x/m))^D}. \tag{4.3}$$

To analyze the inner sum over m, we break it into two pieces depending on whether $m \le x^{1/2}$ or $m > x^{1/2}$:

$$\sum_{\substack{m\le x \\ p|m\Rightarrow p|q, p\nmid r}} \frac{1}{m^\alpha(\log(x/m))^D} \ll \frac{1}{(\log x)^D} \sum_{\substack{m\le x^{1/2} \\ p|m\Rightarrow p|q, p\nmid r}} \frac{1}{m^\alpha} + \sum_{\substack{x^{1/2}<m\le x \\ p|m\Rightarrow p|q, p\nmid r}} \frac{1}{m^\alpha}.$$

To deal with the sum over $x^{1/2} < m \le x$, note that the number of $m \le x$ with $p|m \Rightarrow p|q$ is maximized when q is the product of primes up to $\sim \log x$, in which case the number of such

m is $x^{o(1)}$. Thus the sum over $x^{1/2} < m \le x$ is $O(x^{-\alpha/2+o(1)})$, and their overall contribution to (4.3) is acceptable. To deal with the sum over $m \le x^{1/2}$, note that

$$\sum_{\substack{m\le x^{1/2}\\ p|m\Rightarrow p|q, p\nmid r}} \frac{1}{m^\alpha} \le \sum_{\substack{m\le x^{1/2}\\ m|q,(m,r)=1}} \frac{\mu^2(m)}{\varphi_\alpha(m)},$$

where $\varphi_\alpha(m) = \prod_{p|m}(p^\alpha - 1)$. Thus their overall contribution to (4.3) is

$$\ll \frac{\Psi(x,y)}{(\log x)^D} \sum_{(m,r)=1} \frac{\mu^2(m)}{\varphi_\alpha(m)} \sum_{mr|q\sim Q} \frac{\varphi(r)}{\varphi(q)} \ll \frac{\Psi(x,y)}{(\log x)^D} \sum_m \frac{\mu^2(m)}{\varphi(m)\varphi_\alpha(m)} \ll \frac{\Psi(x,y)}{(\log x)^D},$$

since the infinite sum over m converges. This establishes (4.2) and completes the proof of the lemma. □

To deduce the second part of Theorem 1.2, we apply Proposition 4.2 with $a_q \equiv a_1\overline{a_2} \pmod q$ for each dyadic interval with $Q \le x^{3/5-\varepsilon}$. It is applicable since we assume that the Siegel-Walfisz criterion holds for f with exponent $D \ge A + (6A + 38)$. Summing up over all dyadic intervals, the second part of the Theorem follows (with A replaced by $A - 1$). □

Remark 4.3. One can remove the characters $\psi \pmod{r_\psi}$ with $r_\psi > |\Xi|(\log x)^{A+1}$ from Ξ at an acceptable cost: Let $\mathcal{E}$ be the set of $\psi \pmod{r_\psi} \in \Xi$ with $r_\psi > |\Xi|(\log x)^{A+1}$. Removing $\mathcal{E}$ from Ξ induces an error of at most

$$\sum_{\substack{q\le Q\\ (b,q)=1}} \frac{1}{\varphi(q)} \sum_{\chi\in\mathcal{E}_q} |S_f(x,\chi)|.$$

As $|S_f(x,\chi)| \ll \Psi_q(x,y) \ll (\varphi(q)/q)\Psi(x,y)$ the above is

$$\ll \Psi(x,y) \sum_{\chi\in\mathcal{E}} \sum_{\substack{q\le Q\\ r_\chi|q}} \frac{1}{q} \ll \Psi(x,y)\log x \sum_{\chi\in\mathcal{E}} \frac{1}{r_\chi} \ll \frac{\Psi(x,y)}{(\log x)^A},$$

since

$$\sum_{\chi\in\mathcal{E}} \frac{1}{r_\chi} \le \frac{|\Xi|}{|\Xi|(\log x)^{A+1}} = \frac{1}{(\log x)^{A+1}}.$$

In particular, this shows that we may bound the conductors of the characters in Ξ by $\ll (\log x)^{7A+39}$ in the statement of Theorem 1.2.

5. A large sieve inequality supported on smooth numbers

In this section we prove a large sieve inequality for sequences supported on smooth numbers, a result which may be of independent interest.

Theorem 5.1 (Large sieve for smooth numbers)**.** *There exists $C, c > 0$ such that the following statement holds. Let $(\log x)^C \le y \le x$ be large, and*

$$Q = \min(y^c, \exp(c \log x / \log\log x)).$$

For any sequence $\{a_n\}$ we have

$$\sum_{q \le Q} \sideset{}{^*}\sum_{\chi \pmod q} \left| \sum_{\substack{n \le x \\ P(n) \le y}} a_n \chi(n) \right|^2 \ll \Psi(x,y) \cdot \sum_{\substack{n \le x \\ P(n) \le y}} |a_n|^2.$$

The upper bound is sharp up to a constant, as may be seen by taking each $a_n = 1$, so that the $\chi = 1$ term on the left-hand side equals $\Psi(x,y)^2$, the size of the right-hand side. This result has the advantage over the traditional large sieve inequality that the sequence $\{a_n\}$ is supported on a sparse set (when $y = x^{o(1)}$), but the disadvantage that this inequality holds in a much smaller range for q than the usual $q \ll x^{1/2}$. It may well be that Theorem 5.1 holds with $Q = \Psi(x,y)^{1/2}$.

5.1. **Zero-density estimates.** To prove Theorem 5.1, we will use the following two consequences of deep zero-density results in the literature. The first is a bound for character sums over smooth numbers assuming a suitable zero-free region for the associated L-function (see Section 3 of [7]).

Proposition 5.2. *There is a small positive constant $\delta > 0$ and a large positive constant $\kappa > 0$ such that the following statement holds. Let $(\log x)^{1.1} \le y \le x$ be large. Let $\chi \pmod q$ be a non-principal character with $q \le x$ and conductor $r := \mathrm{cond}(\chi) \le x^\delta$. If $L(s,\chi)$ has no zeros in the region*

$$\mathrm{Re}(s) > 1 - \varepsilon, \quad |\mathrm{Im}(s)| \le T, \tag{5.1}$$

$$\frac{\kappa}{\log y} < \varepsilon \le \frac{\alpha(x,y)}{2}, \quad y^{0.9\varepsilon} (\log x)^2 \le T \le x^\delta, \text{ and } y \ge (Tr)^\kappa. \tag{5.2}$$

Then

$$\left| \sum_{\substack{n \le x \\ P(n) \le y}} \chi(n) \right| \ll \Psi(x,y) \sqrt{(\log x)(\log y)} (x^{-0.3\varepsilon} \log T + T^{-0.02}).$$

We also need the following log-free zero-density estimate by Huxley and Jutila, which can be found in Section 2 of [7]:

Proposition 5.3. *Let $\varepsilon \in [0, 1/2]$, $T \ge 1$ and $Q \ge 1$. Then the function $G_Q(s) = \prod_{q \le Q} \prod^*_{\chi \pmod q} L(s,\chi)$ has $\ll (Q^2 T)^{\frac{5}{2}\varepsilon}$ zeros s, counted with multiplicity, inside the region (5.1).*

5.2. **Proof of Theorem 5.1.** It suffices to prove its dual form:

Proposition 5.4. *There exist $C, c > 0$ such that the following statement holds. Let $(\log x)^C \le y \le x$ be large, and*

$$Q = \min(y^c, \exp(c \log x/\log\log x)).$$

For any sequence $\{b_\chi\}$ we have

$$\sum_{\substack{n\le x\\ P(n)\le y}} \left|\sum_{q\le Q}\ \sideset{}{^*}\sum_{\chi \pmod q} b_\chi \chi(n)\right|^2 \ll \Psi(x,y)\cdot \sum_{q\le Q}\ \sideset{}{^*}\sum_{\chi \pmod q} |b_\chi|^2.$$

Proof. The left hand side can be bounded by

$$\sum_{\chi_1}\sum_{\chi_2} b_{\chi_1}\overline{b_{\chi_2}} \sum_{\substack{n\le x\\ P(n)\le y}} (\chi_1\overline{\chi_2})(n). \tag{5.3}$$

Thus we need to understand character sums over smooth numbers. For $\eta \in (0, 1/2]$, define $\Xi(\eta)$ to be the set of all non-principal characters $\chi \pmod q$ with $q \le Q^2$, such that

$$\eta\Psi(x,y) < \left|\sum_{\substack{n\le x\\ P(n)\le y}} \chi(n)\right| \le 2\eta\Psi(x,y).$$

The contribution to (5.3) from those χ_1, χ_2 with $\chi_1\overline{\chi_2} \in \Xi(\eta)$ is

$$\le 2\eta\Psi(x,y) \sum_{\substack{\chi_1,\chi_2\\ \chi_1\overline{\chi_2}\in\Xi(\eta)}} |b_{\chi_1}b_{\chi_2}| \le 2\eta\Psi(x,y)|\Xi(\eta)| \sum_\chi |b_\chi|^2,$$

using the Cauchy-Schwarz inequality. We will show that $|\Xi(\eta)| \ll \eta^{-1/2}$ so that the result follows by summing over η dyadically. We may assume that $\eta \ge Q^{-8}$, as the bound follows for smaller η from the trivial bound $|\Xi(\eta)| \le Q^4$.

We now use Proposition 5.2 to show that if $\chi \in \Xi(\eta)$ then $L(s,\chi)$ has a zero in the region (5.1) for suitable values of ε and T: Let's assume that $\chi \in \Xi(\eta)$ and $L(s,\chi)$ has no zero in (5.1) with $T = Q^{500}$. We wish to verify the hypotheses in (5.2): The upper bound on T follows from the definition of Q. Now $r \le q \le Q^2$ and so the third hypothesis of (5.2) follows by selecting c so that $502c\kappa \le 1$. We define

$$\varepsilon = \max\left(\frac{2\kappa}{\log y}, \frac{12(\log\eta^{-1} + \log\log x)}{\log x}\right).$$

Since $\log\eta^{-1} \le 8\log Q$ and $\log\log x \le \log Q$, we have the upper bound

$$\varepsilon \le \max\left(\frac{2\kappa}{\log y}, \frac{108\log Q}{\log x}\right) \ll \frac{1}{\log\log x},$$

so that $\varepsilon = o(1)$ and $y^\varepsilon \ll Q^{108}$. The first hypothesis in (5.2) follows immediately. Finally, by selecting C so that $cC \geq 2$ we guarantee that $Q \geq (\log x)^2$, so that the lower bound on T in (5.2) follows easily.

Now $\varepsilon \geq 12(\log \eta^{-1} + \log\log x)/\log x$ so that $x^{-0.3\varepsilon} \leq (\eta/\log x)^{3.6}$, and therefore

$$\sqrt{(\log x)(\log y)}(x^{-0.3\varepsilon}\log T + T^{-0.02}) \leq \eta^{3.6}(\log x)^{-1.6} + Q^{-10}\log x.$$

Now $Q^{-10}\log x = o(Q^{-8}) = o(\eta)$, as $Q \geq (\log x)^2$. Therefore Proposition 5.2 implies that

$$\left|\sum_{\substack{n\leq x\\ P(n)\leq y}} \chi(n)\right| \ll o(\eta\Psi(x,y)),$$

contradicting the definition of $\Xi(\eta)$.

By Proposition 5.3 we now deduce (remembering that $q \leq Q^2$) that

$$|\Xi(\eta)| \ll (Q^4T)^{\frac{5}{2}\varepsilon} = Q^{1260\varepsilon} \leq Q^4 \ll \eta^{-1/2},$$

which completes the proof. □

6. Bounding the number of exceptional characters

Corollary 6.1. *There exist $C, c > 0$ such that the following statement holds. Let $(\log x)^C \leq y \leq x^{1/4}$ be large. Let $\{a_n\}$ be an arbitrary 1-bounded sequence. For $B \geq 0$, let $\Xi(B)$ be the set of primitive characters $\chi \pmod r$ with $r \leq Q$ where*

$$Q := \min(y^c, \exp(c\log x/\log\log x),$$

such that there exists $x^{1/4} < X \leq x$ for which

$$\left|\sum_{\substack{n\leq X\\ P(n)\leq y}} a_n\chi(n)\right| \geq \frac{\Psi(X,y)}{(u\log u)^4(\log x)^B}.$$

Then $|\Xi(B)| \ll (\log x)^{3B+13}$.

Proof. Let $T = (u\log u)^4(\log x)^B$. We begin by partitioning the interval $[x^{1/4}, x]$ using a sequence $x^{1/4} = X_0 < X_1 < \cdots < X_{J-1} < X_J = x$ with $J \asymp T\log x$, such that $X_{j+1} - X_j \asymp \varepsilon X_j/T$, for some fixed small enough $\varepsilon > 0$, for each $0 \leq j < J$.

For each $\chi \in \Xi(B)$, there exists some $0 \leq j < J$ for which

$$\left|\sum_{n\leq X_j} a_n\chi(n)\right| \geq \frac{\Psi(X_j,y)}{T} - \sum_{\substack{X_j<n\leq X_{j+1}\\ P(n)\leq y}} 1.$$

Corollary 2 of [8] implies a good upper bound for smooth numbers in short intervals: For any fixed $\kappa > 0$,

$$\Psi(x + \frac{x}{T}, y) - \Psi(x, y) \ll_\kappa \frac{\Psi(x,y)}{T} \text{ for } 1 \le T \le \min\{y^\kappa, x\}. \tag{6.1}$$

In our case $u \log u \le \log x$ so that $T \le (\log x)^{B+4}$, so the hypothesis here is satisfied, and we therefore have

$$\sum_{\substack{X_j < n \le X_{j+1} \\ P(n) \le y}} 1 = \Psi(X_{j+1}, y) - \Psi(X_j, y) \ll \varepsilon \frac{\Psi(X_j, y)}{T}.$$

By choosing ε sufficiently small we deduce that

$$\left| \sum_{n \le X_j} a_n \chi(n) \right| \ge \frac{\Psi(X_j, y)}{2T}. \tag{6.2}$$

We deduce that there exists some $0 \le j < J$ such that (6.2) holds for at least $|\Xi(B)|/J$ characters $\chi \in \Xi(B)$. Therefore

$$\sum_{\chi \in \Xi(B)} \left| \sum_{\substack{n \le X_j \\ P(n) \le y}} a_n \chi(n) \right|^2 \ge \frac{|\Xi(B)|}{J} \cdot \frac{\Psi(X_j, y)^2}{4T^2} \gg \frac{|\Xi(B)| \Psi(X_j, y)^2}{T^3 \log x}$$

On the other hand, Theorem 5.1 implies that

$$\sum_{\chi \in \Xi(B)} \left| \sum_{\substack{n \le X_j \\ P(n) \le y}} a_n \chi(n) \right|^2 \le \sum_{r \le Q} \sideset{}{^*}\sum_{\chi \pmod r} \left| \sum_{\substack{n \le X_j \\ P(n) \le y}} a_n \chi(n) \right|^2 \ll \Psi(X_j, y)^2,$$

and therefore $|\Xi(B)| \ll T^3 \log x = (u \log u)^{12} (\log x)^{3B+1} \ll (\log x)^{3B+13}$, as claimed. □

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